Lecture notes on p-adic Hodge theory

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Contents

CHAPTER I

Introduction

1. A first glimpse of p-adic Hodge theory

Our goal in this section is to give a brief introduction to p-adic Hodge theory. By nature, p-adic Hodge theory admits two different perspectives, namely the arithmetic one and the geometric one. We illustrate some key ideas of p-adic Hodge theory from each perspective and discuss some fundamental results.

1.1. The arithmetic perspective

A central object in algebraic number theory is the absolute Galois group $\Gamma_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Indeed, $\Gamma_{\mathbb{Q}}$ contains virtually all arithmetic information about the field $\mathbb{Q}$ (and its finite extensions, called *number fields*). However, since $\Gamma_{\mathbb{Q}}$ is an extremely sophisticated object, we usually study it via the natural injective group homomorphism $\Gamma_{\mathbb{Q}_p} \hookrightarrow \Gamma_{\mathbb{Q}}$ induced by the canonical embedding $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$ for each prime p. A general principle is that we can deduce much information about $\Gamma_{\mathbb{Q}}$ from knowledge about $\Gamma_{\mathbb{Q}_p}$ for each prime p.

The group $\Gamma_{\mathbb{Q}_p}$ is still quite complicated but turns out to be much more manageable than the group $\Gamma_{\mathbb{Q}}$ is. The main objective of p-adic Hodge theory, from the arithmetic perspective, is to understand $\Gamma_{\mathbb{Q}_p}$ via continuous representations $\Gamma_{\mathbb{Q}_p} \to \mathrm{GL}_n(\mathbb{Q}_p)$, called *$p$-adic Galois representations*, where $\Gamma_{\mathbb{Q}_p}$ and $\mathrm{GL}_n(\mathbb{Q}_p)$ are respectively endowed with the profinite topology and the p-adic topology. Such representations are particularly interesting as they encode two different kinds of structures on $\mathbb{Q}_p$, namely the algebraic ones from the group $\Gamma_{\mathbb{Q}_p}$ and the analytic ones from the p-adic topology.

In this subsection, we present a primary example that shows why p-adic Galois representations are important for carrying out the strategy outlined in the first paragraph and how we study such representations. Let E be an *elliptic curve* over $\mathbb{Q}$, which refers to a projective curve defined by a polynomial equation

$$y^2 = x^3 + ax + b \quad \text{with } a, b \in \mathbb{Q} \text{ and } 4a^3 + 27b^2 \neq 0. \tag{1.1}$$

Elliptic curves play a fundamental role in modern number theory, as highlighted by the proof of Fermat's last theorem. Elliptic curves have a remarkable property that their points (including the point at infinity) naturally form an abelian group. Hence for each positive integer n and a $\mathbb{Q}$-algebra R, we can define

$$E[n](R) := \{P \in E(R) : nP = O\}$$

where O denotes the point at infinity identified as the zero element in E. We fix a prime ℓ and define the *ℓ-adic Tate module* of E by

$$T_\ell(E) := \varprojlim E[\ell^v](\overline{\mathbb{Q}})$$

where the transition maps send each $P \in E[\ell^{v+1}](\overline{\mathbb{Q}})$ to $\ell P \in E[\ell^v](\overline{\mathbb{Q}})$. It is a standard fact that $T_\ell(E)$ is a free $\mathbb{Z}_\ell$-module of rank 2, thereby admitting an isomorphism

$$T_\ell(E) \simeq \mathbb{Z}_\ell^2.$$

Moreover, the tautological action of $\Gamma_{\mathbb{Q}}$ on $\overline{\mathbb{Q}}$ naturally induces a continuous action on $T_\ell(E)$ and in turn gives rise to a continuous representation of $\Gamma_{\mathbb{Q}}$ on

$$V_\ell(E) := T_\ell(E) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \simeq \mathbb{Q}_\ell^2$$

called the *ℓ-adic rational Tate module* of E. The action of $\Gamma_{\mathbb{Q}}$ on $T_\ell(E)$ and $V_\ell(E)$ contains much information about the elliptic curve E, as suggested by the following fact:

THEOREM 1.1.1 (Faltings [**Fal83**]). Given two elliptic curves E_1 and E_2 over $\mathbb{Q}$, there exist natural isomorphisms

$$\begin{aligned} \operatorname{Hom}(E_1, E_2) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell &\cong \operatorname{Hom}_{\Gamma_{\mathbb{Q}}}(T_\ell(E_1), T_\ell(E_2)), \\ \operatorname{Hom}(E_1, E_2) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell &\cong \operatorname{Hom}_{\Gamma_{\mathbb{Q}}}(V_\ell(E_1), V_\ell(E_2)). \end{aligned} \tag{1.2}$$

In particular, a homomorphism between E_1 and E_2 is uniquely determined by the induced map on the Tate modules as $\Gamma_{\mathbb{Q}}$-representations.

Remark. By a result of Tate [**Tat66**], an analogous statement holds for elliptic curves over $\mathbb{F}_p$ with $p \neq \ell$. Both Theorem 1.1.1 and the result of Tate [**Tat66**] are special cases of the *Tate conjecture* which relates subvarieties of a given algebraic variety X over a field k to representations of $\Gamma_k = \operatorname{Gal}(\overline{k}/k)$ on vector spaces over $\mathbb{Q}_\ell$ that naturally arise from X (similar to the ℓ-adic rational Tate module an elliptic curve). For elliptic curves over $\mathbb{Q}_p$, we get injective maps instead of isomorphisms in (1.2).

However, the action of $\Gamma_{\mathbb{Q}}$ on $T_\ell(E)$ and $V_\ell(E)$ is difficult to understand due to the complexity of the group $\Gamma_{\mathbb{Q}}$. Following the strategy outlined at the beginning of this subsection, we study the action of $\Gamma_{\mathbb{Q}_p}$ on $T_\ell(E)$ and $V_\ell(E)$ for each prime p via the natural injection $\Gamma_{\mathbb{Q}_p} \hookrightarrow \Gamma_{\mathbb{Q}}$. In fact, we have an identification

$$T_\ell(E) \cong \varprojlim E[\ell^v](\overline{\mathbb{Q}}_p) \simeq \mathbb{Z}_\ell^2,$$

endowed with a continuous action of $\Gamma_{\mathbb{Q}_p}$ naturally induced by the tautological action on $\overline{\mathbb{Q}}_p$.

We assume that E has good reduction at p. For $p > 3$, our assumption concretely means that in the polynomial equation (1.1) we have $a, b \in \mathbb{Z}_p$ with $4a^3 + 27b^2$ not divisible by p. The assumption is not very restrictive; indeed, it is a standard fact that E has good reduction at almost all primes (i.e., all but finitely many primes). A main consequence of our assumption is that E admits mod p reduction, denoted by $\overline{E}$, which is an elliptic curve over $\mathbb{F}_p$ with points given by the mod p solutions of (1.1). We have the ℓ-adic Tate module of $\overline{E}$ defined by

$$T_\ell(\overline{E}) := \varprojlim \overline{E}[\ell^v](\overline{\mathbb{F}}_p),$$

which turns out to be a free module over $\mathbb{Z}_\ell$ (but not necessarily of rank 2) with a continuous action of $\Gamma_{\mathbb{F}_p} = \operatorname{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ naturally induced by the tautological action on $\overline{\mathbb{F}}_p$, and consequently obtain a continuous representation of $\Gamma_{\mathbb{F}_p}$ on the ℓ-adic rational Tate module

$$V_\ell(\overline{E}) := T_\ell(\overline{E}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

For $p \neq \ell$, we can explicitly describe the action of $\Gamma_{\mathbb{Q}_p}$ on $T_\ell(E)$ and $V_\ell(E)$ through the action of $\Gamma_{\mathbb{F}_p}$ on $T_\ell(\overline{E})$ and $V_\ell(\overline{E})$. In fact, if we regard $T_\ell(\overline{E})$ and $V_\ell(\overline{E})$ as $\Gamma_{\mathbb{Q}_p}$-representations via the natural surjection $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \operatorname{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p) \cong \Gamma_{\mathbb{F}_p}$, where $\mathbb{Q}_p^{\mathrm{un}}$ denotes the maximal unramified extension of $\mathbb{Q}_p$, we have isomorphisms

$$T_\ell(E) \simeq T_\ell(\overline{E}) \quad \text{and} \quad V_\ell(E) \simeq V_\ell(\overline{E})$$

as $\Gamma_{\mathbb{Q}_p}$-representations. Hence we only need to understand $T_\ell(\overline{E})$ and $V_\ell(\overline{E})$ as (continuous) $\Gamma_{\mathbb{F}_p}$-representations. The group $\Gamma_{\mathbb{F}_p}$ is topologically generated by the Frobenius automorphism which maps each element in $\overline{\mathbb{F}}_p$ to its p-th power. It turns out that the Frobenius automorphism acts on $T_\ell(\overline{E})$ and $V_\ell(\overline{E})$ with characteristic polynomial $x^2 - a_p x + p$, where we set $a_p := p + 1 - \#\overline{E}(\mathbb{F}_p)$. In summary, we can specify the action of $\Gamma_{\mathbb{Q}_p}$ on $T_\ell(E)$ and $V_\ell(E)$ by the following properties:

(i) The action is continuous and factors through the natural surjection $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p}$.

(ii) The Frobenius automorphism of $\overline{\mathbb{F}}_p$, which topologically generates $\Gamma_{\mathbb{F}_p}$, acts with trace $a_p = p + 1 - \#\overline{E}(\mathbb{F}_p)$ and determinant p.

We refer to a $\Gamma_{\mathbb{Q}_p}$-representation with property (i) as an *unramified* representation, motivated by the natural identification $\Gamma_{\mathbb{F}_p} \cong \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{un}}/\mathbb{Q}_p)$. Since the ℓ-adic Tate module $T_\ell(E)$ is unramified, it loses much information about the topology on $\Gamma_{\mathbb{Q}_p}$; indeed, the topology on $\Gamma_{\mathbb{F}_p}$ is very simple (being generated by one element, namely the Frobenius automorphism) compared to the topology on $\Gamma_{\mathbb{Q}_p}$. Intuitively, for $p \neq \ell$ the topologies on $\Gamma_{\mathbb{Q}_p}$ and $\mathbb{Q}_\ell$ do not get along with each other very well, thereby forcing the continuous action of $\Gamma_{\mathbb{Q}_p}$ on $T_\ell(E)$ to be simple. It is worthwhile to mention that our discussion here explains one direction of the following important criterion:

THEOREM 1.1.2 (Néron [**Nér64**], Ogg [**Ogg67**], Shafarevich). An elliptic curve E over $\mathbb{Q}$ has good reduction at p if and only if $T_\ell(E)$ is unramified for a prime $\ell \neq p$.

Let us now set $p = \ell$. We have entered the realm of p-adic Hodge theory, as $V_p(E)$ is a p-adic Galois representation by construction. In stark contrast to our discussion in the previous two paragraphs, we have the following facts:

(1) The (rational) Tate modules for E and $\overline{E}$ are never isomorphic; indeed, $T_p(\overline{E})$ is isomorphic to either $\mathbb{Z}_p$ or 0 whereas $T_p(E)$ is always isomorphic to $\mathbb{Z}_p^2$.

(2) $T_p(E)$ and $V_p(E)$ turn out to be never unramified; in other words, the action of $\Gamma_{\mathbb{Q}_p}$ on $T_p(E)$ and $V_p(E)$ always has a nontrivial contribution from the kernel of the surjection $\Gamma_{\mathbb{Q}_p} \twoheadrightarrow \Gamma_{\mathbb{F}_p}$, called the *inertia group* of $\mathbb{Q}_p$ and denoted by $I_{\mathbb{Q}_p}$.

The second fact indicates that the topologies on $\Gamma_{\mathbb{Q}_p}$ and $\mathbb{Q}_p$ do not clash and thus allow $T_p(E)$ to carry a large amount of topological information. A side effect is that, as the first fact shows, it is impossible to describe $T_p(E)$ solely based on $T_p(\overline{E})$.

We still wish to understand $T_p(E)$ as a $\Gamma_{\mathbb{Q}_p}$-representation using the mod p reduction $\overline{E}$. Following Tate [**Tat66**] and Grothendieck [**Gro71, Gro74**], we regard E as a curve over $\mathbb{Z}_p$ and consider the functors defined by

$$E[p^\infty] := \varinjlim E[p^v] \quad \text{and} \quad \overline{E}[p^\infty] := \varinjlim \overline{E}[p^v],$$

called the *p-divisible groups* of E and $\overline{E}$, where the transition maps are the natural inclusions. For the elliptic curve E, the p-divisible group $E[p^\infty]$ and the Tate module $T_p(E)$ are equivalent objects in the sense that we can determine one from the other. On the other hand, for the mod p reduction $\overline{E}$, the p-divisible group $\overline{E}[p^\infty]$ contains a lot of information that the Tate module $T_p(\overline{E})$ does not; for example, $\overline{E}[p^\infty]$ never vanishes while $T_p(\overline{E})$ often does (as noted in the previous paragraph). Hence the p-divisible groups serve as refinements of the p-adic Tate modules which do not lose too much information under mod p reduction.

A remarkable fact is that we can describe p-divisible groups in terms of linear algebraic objects. A *Dieudonné module* over $\mathbb{Z}_p$ refers to a finite free $\mathbb{Z}_p$-module D equipped with an endomorphism φ_D, called the *Frobenius endomorphism*, such that $\varphi_D(D)$ contains pD. A *Honda system* over $\mathbb{Z}_p$ is a Dieudonné module D over $\mathbb{Z}_p$ together with a submodule $\mathrm{Fil}^1(D)$ such that φ_D induces a natural isomorphism $\mathrm{Fil}^1(D)/p\,\mathrm{Fil}^1(D) \cong D/\varphi_D(D)$.

THEOREM 1.1.3 (Dieudonné [**Die55**], Fontaine [**Fon77**]). Given an elliptic curve E over $\mathbb{Q}$ with good reduction at p, we have the following statements:

(1) The mod p reduction $\overline{E}$ of E functorially gives rise to a Dieudonné module $\mathbb{D}(\overline{E})$ over $\mathbb{Z}_p$ of rank 2, which uniquely determines the isomorphism class of $\overline{E}[p^\infty]$.
(2) For $p > 2$, the elliptic curve E functorially gives rise to a Honda system over $\mathbb{Z}_p$ with underlying Dieudonné module $\mathbb{D}(\overline{E})$, which uniquely determines the isomorphism class of $E[p^\infty]$.

Remark. Let us make some remarks regarding Theorem 1.1.3.

(1) The results of Dieudonné [**Die55**] and Fontaine [**Fon77**] indeed yield anti-equivalences of categories

$$\left\{ \text{ p-divisible groups over } \mathbb{F}_p \right\} \overset{\sim}{\longleftrightarrow} \left\{ \text{ Dieudonné modules over } \mathbb{Z}_p \right\}$$
$$\left\{ \text{ p-divisible groups over } \mathbb{Z}_p \right\} \overset{\sim}{\longleftrightarrow} \left\{ \text{ Honda systems over } \mathbb{Z}_p \right\}$$

where the second anti-equivalence holds only for $p > 2$. For $p = 2$, the second anti-equivalence holds after taking an appropriate subcategory on each side.
(2) The first statement, proved by Dieudonné [**Die55**], was the main motivation for Tate [**Tat66**] and Grothendieck [**Gro71, Gro74**] to study p-divisible groups in relation to the Tate modules, as it suggests that $\overline{E}[p^\infty]$ behaves much as $T_\ell(\overline{E})$ for $p \neq \ell$. The work of Tate [**Tat66**] and Grothendieck [**Gro71, Gro74**] eventually inspired the proof of the second statement by Fontaine [**Fon77**] in an attempt to describe $E[p^\infty]$ via $\mathbb{D}(\overline{E})$ together with some "lifting data".
(3) Our description of Dieudonné modules is potentially misleading. In general, for a Dieudonné module D the endomorphism φ_D should be Frobenius-semilinear in an appropriate sense. For Dieudonné modules over $\mathbb{Z}_p$, however, the Frobenius-semilinearity simply means linearity as the Frobenius automorphism is trivial on the residue field $\mathbb{F}_p$.

Hence for $p > 2$ we can determine the isomorphism class of $T_p(E)$ as a $\Gamma_{\mathbb{Q}_p}$-representation by the Honda system associated to E with underlying Dieudonné module $\mathbb{D}(\overline{E})$. Intuitively, once we fix an element $\sigma \in \Gamma_{\mathbb{Q}_p}$ that lifts the Frobenius automorphism in $\Gamma_{\mathbb{F}_p}$, the Honda system encodes the actions of $I_{\mathbb{Q}_p}$ and σ on $T_p(E)$ respectively by $\mathrm{Fil}^1(\mathbb{D}(\overline{E}))$ and $\varphi_{\mathbb{D}(\overline{E})}$. For $p = 2$, we can still associate a Honda system to E and show that it contains much information about $T_p(E)$, although in general it does not determine the isomorphism class of $T_p(E)$.

If we instead want to study the p-adic Galois representation on $V_p(E)$, we replace the Dieudonné module $\mathbb{D}(\overline{E})$ by $\mathbb{D}(\overline{E}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, called an *isocrystal* over $\mathbb{Q}_p$, which is a finite dimensional vector space over $\mathbb{Q}_p$ equipped with a (Frobenius-semilinear) automorphism. The Honda system associated to E yields the isocrystal $\mathbb{D}(\overline{E}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with the filtration given by the subspace $\mathrm{Fil}^1(\mathbb{D}(\overline{E})) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, called a *filtered isocrystal* over $\mathbb{Q}_p$. Now Theorem 1.1.3 implies for $p > 2$ that the filtered isocrystal associated to E determines the isomorphism class of $V_p(E)$ as a p-adic Galois representation, which turns out to apply also for $p = 2$.

We have thus transferred the study of $T_p(E)$ and $V_p(E)$ as $\Gamma_{\mathbb{Q}_p}$-representations to the study of certain linear algebraic objects, such as Dieudonné modules and isocrystals. In fact, a main theme of p-adic Hodge theory is to construct a dictionary that relates p-adic Galois representations to various linear algebraic objects. Our discussion here illustrates a prototype for such a dictionary.

1.2. The geometric perspective

Our discussion in §1.1 shows how we can study elliptic curves over $\mathbb{Q}$ via their Tate modules as $\Gamma_{\mathbb{Q}}$-representations. It is natural to ask whether we can similarly study other algebraic varieties. Let X be a smooth proper variety over $\mathbb{Q}$. For each $\mathbb{Q}$-algebra R, we write X_R for the base change of X to R. Given an integer $n \geq 0$ and a prime ℓ, we have the *étale cohomology group* $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ which is a finite dimensional vector space over $\mathbb{Q}_\ell$ with a continuous action of $\Gamma_{\mathbb{Q}}$. As a special case, for an elliptic curve E over $\mathbb{Q}$ we have a natural identification

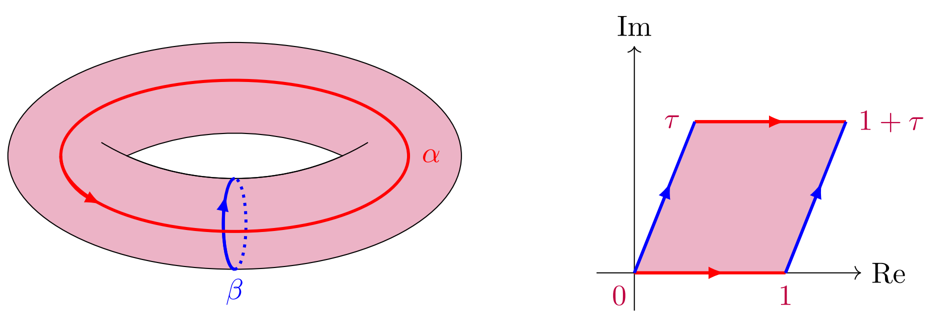
$$V_\ell(E)^\vee \cong H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$$

as $\Gamma_{\mathbb{Q}}$-representations, where $V_\ell(E)^\vee$ denotes the dual representation of $V_\ell(E)$. Following the strategy outlined in §1.1, for each prime p we study the action of $\Gamma_{\mathbb{Q}_p}$ on $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell)$ via the natural injection $\Gamma_{\mathbb{Q}_p} \hookrightarrow \Gamma_{\mathbb{Q}}$; in other words, we study the étale cohomology group $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$ as a representation of $\Gamma_{\mathbb{Q}_p}$. For $p \neq \ell$, the $\Gamma_{\mathbb{Q}_p}$-representation $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_\ell)$ tends to be simple; indeed, it is unramified for all but finitely many $p \neq \ell$, as we have already seen for the rational Tate modules of an elliptic curve in §1.1. For $p = \ell$, on the other hand, $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ as a p-adic Galois representation turns out to carry interesting information about the geometry of X. The main objective of p-adic Hodge theory, from the geometric perspective, is to extract information about the geometric structure of an algebraic variety from the p-adic étale cohomology groups.

In this subsection, we illustrate how the classical Hodge theory inspires fundamental results in p-adic Hodge theory which relates the p-adic étale cohomology groups of an algebraic variety over $\mathbb{Q}_p$ (or its finite extension) to other cohomology groups. Let us consider an elliptic curve E over $\mathbb{Q}$. We may identify $E(\mathbb{C})$ as a complex torus via an isomorphism

$$E(\mathbb{C}) \simeq \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau) \quad \text{for some nonreal } \tau \in \mathbb{C}.$$

Let α and β respectively denote the loops on $E(\mathbb{C})$ induced by the line segments on $\mathbb{C}$ connecting 0 to 1 and τ, as illustrated in the following figure:



We have an isomorphism

$$H_1(E(\mathbb{C}), \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z},$$

with a basis given by the homotopy classes of α and β, and in turn find

$$H^1(E(\mathbb{C}), \mathbb{C}) \cong \operatorname{Hom}(H_1(E(\mathbb{C}), \mathbb{C})) \simeq \mathbb{C} \oplus \mathbb{C} \tag{1.3}$$

by Poincaré duality. Moreover, since $E(\mathbb{C})$ has genus 1, there exists an isomorphism

$$H^0(E_{\mathbb{C}}, \Omega^1_{E_{\mathbb{C}}}) \simeq \mathbb{C}$$

with a basis given by dz. Hence we obtain an isomorphism

$$H^0(E_{\mathbb{C}}, \Omega^1_{E_{\mathbb{C}}}) \oplus \overline{H^0(E_{\mathbb{C}}, \Omega^1_{E_{\mathbb{C}}})} \xrightarrow{\sim} H^1(E(\mathbb{C}), \mathbb{C})$$

sending dz and $d\overline{z}$ respectively to $\int dz = (1, \tau)$ and $\int d\overline{z} = (1, \overline{\tau})$ via the isomorphism (1.3). It is not hard to see that this isomorphism is canonical. In fact, it is a special case of the *Hodge decomposition* given by the following theorem:

THEOREM 1.2.1. For a smooth proper variety X over $\mathbb{C}$, there exists a canonical isomorphism

$$H^n(X(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^n_{\mathrm{dR}}(X/\mathbb{C}) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_X)$$

with $\overline{H^i(X, \Omega^j_X)} = H^j(X, \Omega^i_X)$.

Theorem 1.2.1 admits analogues for the p-adic étale cohomology of an algebraic variety over $\mathbb{Q}_p$. Let $\mathbb{C}_p$ denote the p-adic completion of $\overline{\mathbb{Q}}_p$, called the field of *p-adic complex numbers*. The field $\mathbb{C}_p$ is complete and algebraically closed, just as the field $\mathbb{C}$ is. Since the tautological action of $\Gamma_{\mathbb{Q}_p}$ on $\overline{\mathbb{Q}}_p$ is continuous, it uniquely extends to an action on $\mathbb{C}_p$. For a p-adic analogue of the complex conjugate, we consider the *p-adic cyclotomic character*

$$\chi : \Gamma_{\mathbb{Q}_p} \longrightarrow \mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_p^\times$$

given by the $\Gamma_{\mathbb{Q}_p}$-action on the group

$$T_p(\mu_{p^\infty}) := \varprojlim \mu_{p^v}(\overline{\mathbb{Q}}_p) \simeq \varprojlim \mathbb{Z}/p^v\mathbb{Z} = \mathbb{Z}_p,$$

where $\mu_{p^v}(\overline{\mathbb{Q}}_p)$ denotes the group of p^v-th roots of unity in $\overline{\mathbb{Q}}_p$, and write $\mathbb{C}_p(n)$ for $\mathbb{C}_p$ with $\Gamma_{\mathbb{Q}_p}$-action twisted by χ^n in the sense that each $\gamma \in \Gamma_{\mathbb{Q}_p}$ acts on $\mathbb{C}_p(n)$ as $\chi(\gamma)^n\gamma$. For an elliptic curve E over $\mathbb{Q}_p$ with good reduction, the work of Tate [**Tat67**] yields a canonical isomorphism

$$H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong H^0(E, \Omega^1_{E/\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \oplus H^1(E, \Omega^0_{E/\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(-1)$$

which is compatible with $\Gamma_{\mathbb{Q}_p}$-actions. In fact, this isomorphism is a special case of the *Hodge-Tate decomposition* given by the following theorem:

THEOREM 1.2.2 (Faltings [**Fal88**]). For a smooth proper variety X over $\mathbb{Q}_p$, there exists a canonical isomorphism

$$H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p}) \otimes_{\mathbb{Q}_p} \mathbb{C}_p(-j) \tag{1.4}$$

which is compatible with $\Gamma_{\mathbb{Q}_p}$-actions.

Let us take the *Hodge-Tate period ring* $B_{\mathrm{HT}} := \bigoplus_{n \in \mathbb{Z}} \mathbb{C}_p(n)$ and write the isomorphism (1.4) as a $\Gamma_{\mathbb{Q}_p}$-equivariant isomorphism of graded algebras

$$H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}} \cong \Big(\bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p}) \Big) \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}. \tag{1.5}$$

A result of Tate [**Tat67**] and Sen [**Sen80**] establishes an identification $B_{\mathrm{HT}}^{\Gamma_{\mathbb{Q}_p}} = \mathbb{Q}_p$ and in turn yields an isomorphism of graded $\mathbb{Q}_p$-algebras

$$\big(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}\big)^{\Gamma_{\mathbb{Q}_p}} \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p}).$$

In particular, we can compute the Hodge numbers of X from $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$.

Theorem 1.2.2 is, however, not a complete analogue of Theorem 1.2.1 as it does not give a comparison isomorphism which directly relate the étale cohomology and the de Rham cohomology. Fontaine [**Fon82**] formulated a conjecture that such a comparison isomorphism exists as a refinement of the isomorphism (1.5), inspired by the fact that the de Rham cohomology group $H^n_{\mathrm{dR}}(X/\mathbb{Q}_p)$ has a natural filtration $\left\{ \mathrm{Fil}^m \left(H^n_{\mathrm{dR}}(X/\mathbb{Q}_p) \right) \right\}_{m \in \mathbb{Z}}$, called the *Hodge filtration*, with its graded vector space $\mathrm{gr}\,(H^n_{\mathrm{dR}}(X/\mathbb{Q}_p))$ yielding a natural isomorphism

$$\mathrm{gr}\,(H^n_{\mathrm{dR}}(X/\mathbb{Q}_p)) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p}).$$

A key ingredient of the conjecture is the *de Rham period ring* B_{dR} which Fontaine [**Fon82**] constructed as a $\mathbb{Q}_p$-algebra with the following properties:

(i) B_{dR} carries a natural action of $\Gamma_{\mathbb{Q}_p}$ with $B_{\mathrm{dR}}^{\Gamma_{\mathbb{Q}_p}} = \mathbb{Q}_p$.
(ii) B_{dR} admits a natural filtration $\{ \mathrm{Fil}^n(B_{\mathrm{dR}}) \}_{n \in \mathbb{Z}}$ with B_{HT} as its graded algebra.

Fontaine's conjecture is now a theorem, commonly referred to as the *p-adic de Rham comparison theorem*, which we state as follows:

THEOREM 1.2.3 (Faltings [**Fal89**]). For a smooth proper variety X over $\mathbb{Q}_p$, there exists a canonical isomorphism

$$H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^n_{\mathrm{dR}}(X/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \tag{1.6}$$

which is compatible with $\Gamma_{\mathbb{Q}_p}$-actions and filtrations.

Remark. The filtration on the right side is the *convolution filtration* given by

$$\mathrm{Fil}^m \left(H^n_{\mathrm{dR}}(X/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \right) := \bigoplus_{i+j=m} \mathrm{Fil}^i \left(H^n_{\mathrm{dR}}(X/\mathbb{Q}_p) \right) \otimes_{\mathbb{Q}_p} \mathrm{Fil}^j(B_{\mathrm{dR}}) \quad \text{for every } m \in \mathbb{Z}.$$

Theorem 1.2.3 yields Theorem 1.2.2 as a formal consequence; indeed, we obtain the isomorphism (1.5) from the isomorphism (1.6) by passing to the associated graded vector spaces. In addition, Theorem 1.2.3 induces a natural isomorphism

$$\left(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \right)^{\Gamma_{\mathbb{Q}_p}} \cong H^n_{\mathrm{dR}}(X/\mathbb{Q}_p),$$

thereby allowing us to recover $H^n_{\mathrm{dR}}(X/\mathbb{Q}_p)$ from $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$. Therefore Theorem 1.2.3 (with Theorem 1.2.2 as its consequence) indicates that the p-adic étale cohomology of an algebraic variety over $\mathbb{Q}_p$ behaves much as the singular cohomology of an algebraic variety over $\mathbb{C}$ does.

Let us now assume that X has good reduction over $\mathbb{Q}_p$. Intuitively, our assumption means that we may regard X as a smooth scheme over $\mathbb{Z}_p$ and thus allows us to take its mod p reduction $\overline{X}$. Motivated by our discussion in §1.1, we wish to understand the p-adic Galois representation $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ using $\overline{X}$. We consider the *crystalline cohomology group* $H^n_{\mathrm{cris}}(\overline{X}/\mathbb{Z}_p)$ which is a Dieudonné module over $\mathbb{Z}_p$ with a natural isomorphism

$$H^n_{\mathrm{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^n_{\mathrm{dR}}(X/\mathbb{Q}_p)$$

and a canonical filtration $\left\{ \mathrm{Fil}^m \left(H^n_{\mathrm{cris}}(\overline{X}/\mathbb{Z}_p) \right) \right\}_{m \in \mathbb{Z}}$ induced by the Hodge filtration on $H^n_{\mathrm{dR}}(X/\mathbb{Q}_p)$. For an elliptic curve E with good reduction over $\mathbb{Q}_p$, we may naturally identify $H^1_{\mathrm{cris}}(\overline{E}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with the filtered isocrystal associated to E, which in turn determines $H^1_{\text{ét}}(E_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \cong V_p(E)^\vee$ by our discussion in §1.1. For the general case, Grothendieck [**Gro71**] proposed a conjecture that $H^n_{\mathrm{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ as a filtered isocrystal determines $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ as a p-adic Galois representation in a functorial way; indeed, his conjecture predicts that there exists a fully faithful functor $\mathcal{D}$ on a certain category of p-adic Galois representations with

$$\mathcal{D}\big(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)\big) = H^n_{\mathrm{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

We refer to the functor $\mathcal{D}$ as the *Grothendieck mysterious functor*.

Fontaine [**Fon82, Fon83**] reformulated the conjecture of Grothendieck [**Gro71**] in terms of a comparison isomorphism between the étale cohomology and the crystalline cohomology. His idea was to refine the de Rham comparison isomorphism (1.6) by constructing the *crystalline period ring* B_{cris}, which is a $\mathbb{Q}_p$-subalgebra of B_{dR} with the following properties:

(i) B_{cris} carries a natural $\Gamma_{\mathbb{Q}_p}$-action with $B_{\text{cris}}^{\Gamma_{\mathbb{Q}_p}} = \mathbb{Q}_p$, induced by the action on B_{dR}.
(ii) B_{cris} admits a natural filtration $\{\, \text{Fil}^n(B_{\text{cris}})\,\}_{n\in\mathbb{Z}}$ given by the filtration on B_{dR}.
(iii) B_{cris} contains the maximal unramified extension $\mathbb{Q}_p^{\text{un}}$ of $\mathbb{Q}_p$ with a canonical extension of the Frobenius automorphism on $\mathbb{Q}_p^{\text{un}}$, called the *Frobenius endomorphism.*

Fontaine's conjecture is now a theorem, commonly referred to as the *crystalline comparison theorem*, which we state as follows:

Theorem 1.2.4 (Faltings [**Fal89**]). For a smooth proper variety X over $\mathbb{Q}_p$ with mod p reduction $\overline{X}$, there exists a canonical isomorphism

$$H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}} \cong H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}} \tag{1.7}$$

which is compatible with $\Gamma_{\mathbb{Q}_p}$-actions, filtrations, and Frobenius endomorphisms.

Remark. As in Theorem 1.2.3, the right side carries the convolution filtration given by

$$\text{Fil}^m\left(H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}}\right) := \bigoplus_{i+j=m} \text{Fil}^i\left(H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p)\right) \otimes_{\mathbb{Z}_p} \text{Fil}^j(B_{\text{cris}}) \quad \text{for every } m \in \mathbb{Z}.$$

Under the assumption that X has good reduction, we can obtain the de Rham comparison isomorphism (1.6) from the crystalline comparison isomorphism (1.7) by tensoring with B_{dR} and forgetting the Frobenius endomorphisms. Moreover, Theorem 1.2.4 yields a natural isomorphism

$$\left(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{cris}}\right)^{\Gamma_{\mathbb{Q}_p}} \cong H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p,$$

thereby suggesting that the mysterious functor $\mathcal{D}$ takes the form

$$\mathcal{D}(V) = (V \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_{\mathbb{Q}_p}}$$

for every p-adic Galois representation V. It turns out, by the work of Fontaine [**Fon94b**], that the functor $\mathcal{D}$ is fully faithful on a suitable category of p-adic Galois representations with values taken in the category of filtered isocrystals. In fact, $H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ determines $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ by an identification

$$H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \cong \left(H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}}\right)^{\varphi=1} \cap \text{Fil}^0\left(H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}}\right) \tag{1.8}$$

where we denote by φ the natural Frobenius action on $H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\text{cris}}$ induced by the Frobenius endomorphisms on $H^n_{\text{cris}}(\overline{X}/\mathbb{Z}_p)$ and B_{cris}.

As our discussion demonstrates, a main theme in p-adic Hodge theory is to establish a comparison isomorphism that relates p-adic étale cohomology groups to cohomology groups of a different kind. In addition to the theorems presented in this subsection, there are many results of a similar flavor, notably by the work of Tsuji [**Tsu99**], Scholze [**Sch13**], and Bhatt-Morrow-Scholze [**BMS18, BMS19**]. Let us also mention that there are other approaches for the comparison theorems presented in this subsection, in particular by the work of Fontaine-Messing [**FM87**], Nizioł [**Niz98, Niz08**], and Beilinson [**Bei12, Bei13**].

2. Machinery of p-adic Hodge theory

Our main objective for this section is to present some central tools for p-adic Hodge theory. We demonstrate how these tools provide systemic ways to study p-adic Galois representations and related objects. In addition, we illustrate some of their key properties and applications.

2.1. Period rings and their associated functors

In this subsection, we describe a connection between the two main themes of p-adic Hodge theory provided by some linear algebraic functors. These functors originate in the work of Fontaine [**Fon79, Fon82, Fon83, Fon94a**] which proposes a uniform approach for the p-adic comparison theorems in an attempt to resolve the conjecture of Grothendieck [**Gro71**] on the mysterious functor. We write $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$ for the category of p-adic Galois representations and $\mathrm{Vect}_{\mathbb{Q}_p}$ for the category of vector spaces over $\mathbb{Q}_p$. Let B be a *p-adic period ring*, such as B_{HT}, B_{dR} or B_{cris}, which is a $\mathbb{Q}_p$-algebra carrying a natural $\Gamma_{\mathbb{Q}_p}$-action with $B^{\Gamma_{\mathbb{Q}_p}} = \mathbb{Q}_p$. We define the functor $D_B : \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p}) \longrightarrow \mathrm{Vect}_{\mathbb{Q}_p}$ by setting

$$D_B(V) := (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_{\mathbb{Q}_p}} \quad \text{for each } V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$$

and say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$ is *B-admissible* if the natural $\Gamma_{\mathbb{Q}_p}$-equivariant map

$$\alpha_V : D_B(V) \otimes_{\mathbb{Q}_p} B \longrightarrow (V \otimes_{\mathbb{Q}_p} B) \otimes_{\mathbb{Q}_p} B \cong V \otimes_{\mathbb{Q}_p} (B \otimes_{\mathbb{Q}_p} B) \longrightarrow V \otimes_{\mathbb{Q}_p} B$$

is an isomorphism. We enhance the functor D_B by incorporating additional structures on B, as demonstrated by the following examples:

(1) $D_{B_{\mathrm{HT}}}(V)$ for each $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$ carries a grading naturally induced by the grading on B_{HT}.
(2) $D_{B_{\mathrm{dR}}}(V)$ for each $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$ carries a filtration naturally induced by the filtration on B_{dR}.
(3) $D_{B_{\mathrm{cris}}}(V)$ for each $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_{\mathbb{Q}_p})$ carries a Frobenius endomorphism and a filtration naturally induced by the ones on B_{cris}.

For every smooth proper variety X over $\mathbb{Q}_p$, we may state the p-adic comparison theorems from §1.2 as follows:

(1) $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is B_{HT}-admissible with a natural isomorphism

$$D_{B_{\mathrm{HT}}}(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p})$$

which is compatible with the gradings on both sides.

(2) $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is B_{dR}-admissible with a natural isomorphism

$$D_{B_{\mathrm{dR}}}(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)) \cong H^n_{\mathrm{dR}}(X/\mathbb{Q}_p)$$

which is compatible with the filtrations on both sides.

(3) If X admits mod p reduction $\overline{X}$, then $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is B_{cris}-admissible with a natural isomorphism

$$D_{B_{\mathrm{cris}}}(H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)) \cong H^n_{\mathrm{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

which is compatible with the Frobenius endomorphisms and filtrations on both sides.

Moreover, the notion of B_{cris}-admissibility yields a p-adic analogue of Theorem 1.1.2.

Theorem 2.1.1 (Coleman-Iovita [**CI99**], Breuil [**Bre00**]). An elliptic curve E over $\mathbb{Q}$ has good reduction at p if and only if $V_p(E)$ is B_{cris}-admissible.

Let us denote by $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ the category of B-admissible representations. The work of Fontaine [**Fon82, Fon83**] yields a hierarchy of p-adic Galois representations given by

$$\mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_{\mathbb{Q}_p}) \subsetneq \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{dR}}}(\Gamma_{\mathbb{Q}_p}) \subsetneq \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{HT}}}(\Gamma_{\mathbb{Q}_p})$$

with the associated functors satisfying the following relations:

- $D_{B_{\mathrm{HT}}}(V)$ for each $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{dR}}}(\Gamma_{\mathbb{Q}_p})$ is naturally isomorphic to the graded vector space of $D_{B_{\mathrm{dR}}}(V)$.
- $D_{B_{\mathrm{dR}}}(V)$ for each $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_{\mathbb{Q}_p})$ is naturally isomorphic to $D_{B_{\mathrm{cris}}}(V)$ (after forgetting the Frobenius endomorphism).

This hierarchy realizes relations between various cohomology groups for a smooth proper variety X over $\mathbb{Q}_p$, as presented in §1.2 and summarized in the following statements:

- The Hodge-Tate decomposition (1.5) follows from the de Rham comparison isomorphism (1.6) by passing to the associated graded space via the identification
$$\mathrm{gr}\,(H^n_{\mathrm{dR}}(X/\mathbb{Q}_p)) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/\mathbb{Q}_p}).$$
where $\mathrm{gr}\,(H^n_{\mathrm{dR}}(X/\mathbb{Q}_p))$ denote the graded vector space of $H^n_{\mathrm{dR}}(X/\mathbb{Q}_p)$.
- If X has good reduction, the de Rham comparision isomorphism (1.6) follows from the crystalline comparison isomorphism (1.7) by tensoring with B_{dR} via the identification
$$H^n_{\mathrm{cris}}(\overline{X}/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong H^n_{\mathrm{dR}}(X/\mathbb{Q}_p).$$

We wish to understand how the category $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ behaves, especially in conjunction with the functor D_B. A general formalism developed by Fontaine [**Fon94b**] shows that $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ and D_B have the following properties:

(i) D_B is exact and faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$.

(ii) $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ is closed under taking subquotients.

(iii) $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ is closed under tensor products, with a natural identification
$$D_B(V \otimes_{\mathbb{Q}_p} W) \cong D_B(V) \otimes_{\mathbb{Q}_p} D_B(W) \quad \text{for any } V, W \in \mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p}).$$

(iv) $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$ is closed under taking duals, with a natural identification
$$D_B(V^\vee) \cong D_B(V)^\vee \quad \text{for every } V \in \mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_{\mathbb{Q}_p})$$
where $V^\vee$ and $D_B(V)^\vee$ respectively denote the duals of V and $D_B(V)$.

Moreover, $D_{B_{\mathrm{cris}}}$ and $\mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_{\mathbb{Q}_p})$ have a remarkable property given by the following result:

Theorem 2.1.2 (Fontaine [**Fon94b**]). *The functor $D_{B_{\mathrm{cris}}}$ is fully faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_{\mathbb{Q}_p})$.*

Our discussion in this subsection indicates that period rings and their associated functors provide a general framework for the two main themes in p-adic Hodge theory. From the arithmetic perspective, they provide dictionaries for classifying and studying p-adic Galois representations in terms of linear algebraic objects. From the geometric perspective, they allow us to uniformly formulate p-adic comparison theorems and to systemically detect geometric properties of an algebraic variety over $\mathbb{Q}_p$ from its p-adic étale cohomology. Therefore period rings and their associated functors are essential for studying p-adic Hodge theory via the interplay between the arithmetic and geometric perspectives.

2.2. The Fargues-Fontaine curve and its vector bundles

In this subsection, we provide a brief introduction to a remarkable geometric object called the *Fargues-Fontaine curve*, which plays a fundamental role in modern p-adic Hodge theory. We exhibit its key properties in comparison to the complex projective line $\mathbb{P}^1_{\mathbb{C}}$. In addition, we describe how it provides a geometric framework for studying p-adic Galois representations via its vector bundles (i.e., locally free sheaves of finite rank).

Let us recall that $\mathbb{P}^1_{\mathbb{C}}$ has the following properties:

(i) It is noetherian, connected, and regular of dimension 1.
(ii) Its Picard group $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}})$ is canonically isomorphic to $\mathbb{Z}$.
(iii) It has arithmetic genus 0 in the sense that $H^1(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}})$ vanishes.
(iv) It admits a closed point ∞, namely the point at infinity, with natural isomorphisms

$$\mathbb{P}^1_{\mathbb{C}} - \infty \cong \mathrm{Spec}\,(\mathbb{C}[z]) \quad \text{and} \quad \widehat{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}},\infty}} \cong \mathbb{C}[[z^{-1}]]$$

where $\widehat{\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}},\infty}}$ denotes the completed local ring at ∞.

Property (iv) is essentially a geometric formulation of the natural exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathbb{C}[z] \longrightarrow \mathbb{C}((z^{-1}))/\mathbb{C}[[z^{-1}]] \longrightarrow 0. \tag{2.1}$$

Intuitively, this exact sequence indicates that we can construct $\mathbb{P}^1_{\mathbb{C}}$ by gluing the complex affine line $\mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\,(\mathbb{C}[z])$ to the infinitesimal disk at ∞, given by $\mathrm{Spec}\,(\mathbb{C}[[z^{-1}]])$, along the punctured infinitesimal disk at ∞, given by $\mathrm{Spec}\,(\mathbb{C}((z^{-1})))$.

The construction of the Fargues-Fontaine curve stems from a remarkable discovery of Fontaine [**Fon94a**] that the exact sequence (2.1) admits an analogue for p-adic period rings. By construction, the de Rham period ring B_{dR} is a discretely valued complete field with residue field $\mathbb{C}_p$. We write B^+_{dR} for the valuation ring of B_{dR} and $B_e := B_{\mathrm{cris}}^{\varphi=1}$ for the ring of φ-invariants in B_{cris}, where φ denotes the Frobenius endomorphism on B_{cris}.

Theorem 2.2.1 (Fontaine [**Fon94a**]). There exists a natural exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow B_{\mathrm{dR}}/B^+_{\mathrm{dR}} \longrightarrow 0. \tag{2.2}$$

The exact sequences (2.1) and (2.2) have the following similarities:

(1) $\mathbb{C}[[z^{-1}]]$ and B^+_{dR} are both complete discrete valuation rings, with fraction fields respectively given by $\mathbb{C}((z^{-1}))$ and B_{dR}.
(2) $\mathbb{C}[z]$ and B_e are both principal ideal domains.

The second similarity is another surprising discovery of Fontaine, primarily based on the work of Berger [**Ber08**]. The similarities of the exact sequences (2.1) and (2.2) inspire the construction of the Fargues-Fontaine curve X by gluing $\mathrm{Spec}\,(B_e)$ and $\mathrm{Spec}\,(B^+_{\mathrm{dR}})$ along $\mathrm{Spec}\,(B_{\mathrm{dR}})$.

Theorem 2.2.2 (Fargues-Fontaine [**FF18**]). The Fargues-Fontaine curve X is a $\mathbb{Q}_p$-scheme with the following properties:

(i) It is noetherian, connected and regular of dimension 1.
(ii) Its Picard group $\mathrm{Pic}(X)$ is canonically isomorphic to $\mathbb{Z}$.
(iii) It has arithmetic genus 0 in the sense that $H^1(X, \mathcal{O}_X)$ vanishes.
(iv) It admits a closed point ∞ with natural isomorphisms

$$X - \infty \cong \mathrm{Spec}\,(B_e) \quad \text{and} \quad \widehat{\mathcal{O}_{X,\infty}} \cong B^+_{\mathrm{dR}}$$

where $\widehat{\mathcal{O}_{X,\infty}}$ denotes the completed local ring at ∞.

For an explicit description of the Fargues-Fontaine curve, we have a natural isomorphism $X \cong \operatorname{Proj}(P)$ for a graded ring

$$P := \bigoplus_{n \geq 0} B_e^{(n)}$$

where we set $B_e^{(n)} := \{ f \in B_e : \nu_\infty(f) \geq -n \}$ with ν_∞ denoting the valuation on B_{dR}. For comparison, we have the identification $\mathbb{P}^1_{\mathbb{C}} = \operatorname{Proj}(\mathbb{C}[z_0, z_1])$ and an isomorphism

$$\mathbb{C}[z_0, z_1] \cong \bigoplus_{n \geq 0} \mathbb{C}[z]^{(n)}$$

where we set $\mathbb{C}[z]^{(n)} := \{ f \in \mathbb{C}[z] : \nu_\infty(f) \geq -n \} = \{ f \in \mathbb{C}[z] : \deg(f) \leq n \}$ with ν_∞ denoting the valuation on $\mathbb{C}((z^{-1}))$. The graded rings P and $\mathbb{C}[z_0, z_1]$ have an important common feature of being generated in degree 1 (i.e., being generated by elements in $B_e^{(1)}$ and $\mathbb{C}[z]^{(1)}$).

However, unlike $\mathbb{P}^1_{\mathbb{C}}$, the Fargues-Fontaine curve is not an algebraic variety. The main issue is that it is not of finite type over the base field $\mathbb{Q}_p$. In fact, Theorem 2.2.2 shows that the residue field at ∞ is $\mathbb{C}_p$ and thus is not a finite extension of $\mathbb{Q}_p$.

The work of Fargues-Fontaine [**FF18**] reveals a tidy connection between the category Bun_X of vector bundles on X and the category $\varphi\text{−}\mathrm{Mod}_{\mathbb{Q}_p}$ of isocrystals over $\mathbb{Q}_p$, given by an essentially surjective functor

$$\mathcal{E} : \varphi\text{−}\mathrm{Mod}_{\mathbb{Q}_p} \longrightarrow \mathrm{Bun}_X \,.$$

The key fact is that we can produce a vector bundle $\mathcal{V}$ on X by gluing a vector bundle $\mathcal{V}_e$ on $\operatorname{Spec}(B_e)$ to a vector bundle $\widehat{\mathcal{V}_\infty}$ on $\operatorname{Spec}(B_{\mathrm{dR}}^+)$ along $\operatorname{Spec}(B_{\mathrm{dR}})$; in other words, we obtain a vector bundle on X from a pair (M_e, M_{dR}^+) consisting of a free B_e-module M_e of finite rank and a B_{dR}^+-lattice M_{dR}^+ in $M_e \otimes_{B_e} B_{\mathrm{dR}}$. The functor $\mathcal{E}$ sends each isocrystal D over $\mathbb{Q}_p$ to the vector bundle obtained from $((D \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\varphi=1}, D \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)$, where φ denotes the natural Frobenius action on $D \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$ induced by the Frobenius endomorphisms on D and B_{cris}.

On the category $\mathrm{MF}^{\varphi}_{\mathbb{Q}_p}$ of filtered isocrystals over $\mathbb{Q}_p$, we have another functor

$$\mathcal{F} : \mathrm{MF}^{\varphi}_{\mathbb{Q}_p} \longrightarrow \mathrm{Bun}_X$$

which sends each filtered isocrystal D over $\mathbb{Q}_p$ with filtration $\{ \mathrm{Fil}^n(D) \}_{n \in \mathbb{Z}}$ to the vector bundle obtained from the pair $((D \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\varphi=1}, \mathrm{Fil}^0(D \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}))$ with

$$\mathrm{Fil}^0(D \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Fil}^n(D) \otimes_{\mathbb{Q}_p} \mathrm{Fil}^{-n}(B_{\mathrm{dR}}).$$

The vector bundle $\mathcal{F}(D)$ for each $D \in \mathrm{MF}^{\varphi}_{\mathbb{Q}_p}$ carries a natural action of $\Gamma_{\mathbb{Q}_p}$ induced by the $\Gamma_{\mathbb{Q}_p}$-action on B_{dR}, as the ring B_e and the filtration on B_{dR} turn out to be stable under the $\Gamma_{\mathbb{Q}_p}$-action on B_{dR}. The functor $\mathcal{F}$ allows us to study filtered isocrystals and p-adic Galois representations via vector bundles on X, as indicated by the following facts:

(1) There exists a natural $\Gamma_{\mathbb{Q}_p}$-equivariant isomorphism

$$V \cong H^0(X, \mathcal{F}(D_{B_{\mathrm{cris}}}(V))) \quad \text{ for every } V \in \mathrm{Rep}_{\mathbb{Q}_p}^{B_{\mathrm{cris}}}(\Gamma_{\mathbb{Q}_p}).$$

(2) Every $D \in \mathrm{MF}^{\varphi}_{\mathbb{Q}_p}$ lies in the essential image of $D_{B_{\mathrm{cris}}}$ if and only if $\mathcal{F}(D)$ is trivial.

It is worthwhile to mention that applications of the Fargues-Fontaine curve reach far beyond p-adic Hodge theory. In fact, the seminal work of Fargues-Scholze [**FS21**] shows that the Fargues-Fontaine curve provides powerful geometric tools for studying ℓ-adic Galois representations in relation to algebraic groups. The book of Scholze-Weinstein [**SW20**] is a wonderful introductory reference for the theoretical foundations of these applications.

Exercises

1. Let E be the projective curve over $\mathbb{Q}$ given by the equation $y^2 = x^3 + x$.

(1) Show that E is an elliptic curve over $\mathbb{Q}$ with good reduction at all odd primes.

(2) Give an explicit description of the group law and the $\Gamma_{\mathbb{Q}}$-action on $E[2](\overline{\mathbb{Q}})$.

(3) Give an explicit description of the group law and the $\Gamma_{\mathbb{F}_5}$-action on $\overline{E}[2](\overline{\mathbb{F}_5})$, where $\overline{E}$ denotes the mod-5 reduction of E.

2. Let M be a 2×2 matrix over $\mathbb{Z}_p$.

(1) When M has all entries in $p\mathbb{Z}_p$, prove that there exists a Dieudonné module D over $\mathbb{Z}_p$ of rank 2 with φ_D represented by M if and only if we have $\det(M) \notin p^3\mathbb{Z}_p$.

(2) When M has an entry in $\mathbb{Z}_p^\times$, prove that there exists a Dieudonné module D over $\mathbb{Z}_p$ of rank 2 with φ_D represented by M if and only if we have $\det(M) \notin p^2\mathbb{Z}_p$.

Hint. Consider the Smith normal form of M.

3. In this exercise, we provide a simple analogy between the complex conjugation and the p-adic cyclotomic character.

(1) Show that the complex conjugation naturally induces a character

$$\tilde{\chi} : \Gamma_{\mathbb{R}} \longrightarrow \mathrm{Aut}(\mathbb{R}) \cong \mathbb{R}^\times$$

with $\gamma(\zeta) = \zeta^{\tilde{\chi}(\gamma)}$ for every $\gamma \in \Gamma_{\mathbb{R}}$ and $\zeta \in \mu_\infty$, where μ_∞ denotes the group of roots of unity in $\mathbb{C}$.

(2) Show that the p-adic cyclotomic character χ yields the relation $\gamma(\zeta) = \zeta^{\chi(\gamma)}$ for every $\gamma \in \Gamma_{\mathbb{Q}_p}$ and $\zeta \in \mu_{p^\infty}$, where μ_{p^∞} denotes the group of p-power roots of unity in $\overline{\mathbb{Q}}_p$.

4. Let B be a p-adic period ring and $\eta : \Gamma_{\mathbb{Q}_p} \to \mathbb{Q}_p^\times$ be a continuous character such that the induced p-adic Galois representation is B-admissible.

(1) Prove that there exists an element $b \in B$ with $\gamma(b) = \eta(\gamma)^{-1}b$ for every $\gamma \in \Gamma_{\mathbb{Q}_p}$.

(2) If η is not trivial, prove that b is transcendental over $\mathbb{Q}_p$.

Hint. Take an element $\gamma \in \Gamma_{\mathbb{Q}_p}$ with $\eta(\gamma) \neq 1$. If b satisfies a polynomial equation of degree d over $\mathbb{Q}_p$, we can apply the action of γ to see that b satisfies a polynomial equation of degree $d-1$ over $\mathbb{Q}_p$.

5. Let A and B be subrings of a valued field C with valuation ring $\mathcal{O}_C$ and an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C/\mathcal{O}_C \longrightarrow 0.$$

(1) If A is a field, show that the valuation of each $f \in B$ is nonnegative.

(2) For every $f, g \in B$ with $g \neq 0$, show that there exist elements $a, b \in B$ with

$$f = ag + b \quad \text{and} \quad -\nu(b) \leq -\nu(g),$$

where ν denotes the valuation on C.

Remark. This exercise and Theorem 2.2.1 together imply that B_e admits an almost Euclidean function given by the valuation of B_{dR}.

CHAPTER II

Foundations of p-adic Hodge theory

1. Finite flat group schemes

In this section, we develop basic theory of finite flat group schemes and discuss some of its applications to arithmetic geometry. Our primary reference for this section is the article of Tate [**Tat97**]. Throughout our discussion, all rings are commutative unless specified otherwise.

1.1. Basic definitions and properties

We begin with the notion of group schemes over a base scheme S. We usually take S to be affine and denote the base ring by R.

Definition 1.1.1. A *group scheme* over S, or an S*-group*, is an S-scheme G with maps

- $m : G \times_S G \to G$, called the *multiplication*,
- $e : S \to G$, called the *unit section*,
- $i : G \to G$, called the *inverse*,

which satisfy the group axioms given by the following commutative diagrams:

(a) associativity diagram

$$\begin{array}{ccc} G \times_S G \times_S G & \xrightarrow{m\times \mathrm{id}} & G \times_S G \\ \downarrow{\scriptstyle \mathrm{id}\times m} & & \downarrow{\scriptstyle m} \\ G \times_S G & \xrightarrow{\quad m \quad} & G \end{array}$$

(b) identity diagrams

$$\begin{array}{ccccc} G \times_S S & & \xrightarrow{\quad\sim\quad} & & G \\ & {\scriptstyle \mathrm{id}\times e}\searrow & & \nearrow{\scriptstyle m} & \\ & & G \times_S G & & \end{array} \qquad \begin{array}{ccccc} S \times_S G & & \xrightarrow{\quad\sim\quad} & & G \\ & {\scriptstyle e\times \mathrm{id}}\searrow & & \nearrow{\scriptstyle m} & \\ & & G \times_S G & & \end{array}$$

(c) inverse diagram

$$\begin{array}{ccc} G & \underset{(\mathrm{id},i)}{\overset{(i,\mathrm{id})}{\rightrightarrows}} & G \times_S G \\ \downarrow & & \downarrow{\scriptstyle m} \\ S & \xrightarrow{\quad e \quad} & G \end{array}$$

Remark. In other words, S-groups are group objects in the category of S-schemes.

LEMMA 1.1.2. A scheme G over S is a group scheme if and only if it defines a functor from the category of S-schemes to the category of groups sending each S-scheme T to $G(T)$.

PROOF. The assertion is evident by Yoneda's lemma. □

Definition 1.1.3. Let $f : G \to H$ be an S-scheme morphism between S-groups G and H.

(1) We say that f is a *homomorphism* if the induced map $f_T : G(T) \to H(T)$ for each S-scheme T is a group homomorphism.
(2) If f is a homomorphism, we define its *kernel* to be the functor $\ker(f)$ which maps each S-scheme T to the kernel of the induced map $f_T : G(T) \to H(T)$.

Example 1.1.4. Given an S-group G and an integer n, the *multiplication by* n on G is the homomorphism $[n]_G : G \to G$ given by the n-th power map on $G(T)$ for each S-scheme T.

Remark. The homomorphisms $[-1]_G$ and $[1]_G$ respectively coincide with the inverse map and the identity map of G.

LEMMA 1.1.5. Given an S-group homomorphism $f : G \to H$, its kernel is an S-group and is naturally isomorphic to the fiber of f over the unit section of H.

PROOF. The assertion is straightforward to verify by Lemma 1.1.2. □

Definition 1.1.6. Let $G = \mathrm{Spec}\,(A)$ be an affine R-group.

(1) Its *comultiplication* is the map $\mu : A \to A \otimes_R A$ induced by the multiplication.
(2) Its *counit* is the map $\epsilon : A \to R$ induced by the unit section.
(3) Its *coinverse* is the map $\iota : A \to A$ induced by the inverse.

Example 1.1.7. We present some important examples of affine R-groups.

(1) The *additive R-group* is the R-scheme $\mathbb{G}_a := \mathrm{Spec}\,(R[t])$ with the natural additive group structure on $\mathbb{G}_a(B) = B$ for each R-algebra B. Its comultiplication μ, counit ϵ, and coinverse ι are the R-algebra homomorphisms with
$$\mu(t) = t \otimes 1 + 1 \otimes t, \quad \epsilon(t) = 0, \quad \iota(t) = -t.$$
(2) The *multiplicative R-group* is the R-scheme $\mathbb{G}_m := \mathrm{Spec}\,(R[t, t^{-1}])$ with the natural multiplicative group structure on $\mathbb{G}_m(B) = B^\times$ for each R-algebra B. Its comultiplication μ, counit ϵ, and coinverse ι are the R-algebra homomorphisms with
$$\mu(t) = t \otimes t, \quad \epsilon(t) = 1, \quad \iota(t) = t^{-1}.$$
(3) The *n-th roots of unity* for $n \geq 1$ is the R-scheme $\mu_n := \mathrm{Spec}\,(R[t]/(t^n - 1))$ with the natural multiplicative group structure on $\mu_n(B) = \{\, b \in B : b^n = 1 \,\}$ for each R-algebra B. We can regard μ_n as a closed R-subgroup of $\mathbb{G}_m$ via the natural surjection $R[t, t^{-1}] \twoheadrightarrow R[t]/(t^n - 1)$ with the induced comultiplication, counit, and coinverse.
(4) If R has characteristic p, the *Frobenius kernel* is the R-scheme $\alpha_p := \mathrm{Spec}\,(R[t]/t^p)$ with the natural additive group structure on $\alpha_p(B) = \{\, b \in B : b^p = 0 \,\}$ for each R-algebra B. We can regard α_p as a closed R-subgroup of $\mathbb{G}_a$ via the natural surjection $R[t] \twoheadrightarrow R[t]/(t^p)$ with the induced comultiplication, counit, and coinverse.
(5) Given an abstract group M, the *constant R-group* associated to M is the R-scheme $\underline{M} := \coprod_{m \in M} \mathrm{Spec}\,(R) \cong \mathrm{Spec}\,(A)$ for $A := \prod_{m \in M} R$ with the natural group structure induced by M on $\underline{M}(B)$ for each R-algebra B, regarded as the set of locally constant functions from $\mathrm{Spec}\,(B)$ to M. If we identify A and $A \otimes_R A$ respectively as the rings of R-valued functions on M and $M \times M$, we can describe the comultiplication μ, counit ϵ, and coinverse ι by the equalities
$$\mu(f)(m, m') = f(mm'), \quad \epsilon(f) = f(1_M), \quad \iota(f)(m) = f(m^{-1})$$
for each $f \in A$ and $m, m' \in M$, where 1_M denotes the identity element of M.

LEMMA 1.1.8. Let $G = \operatorname{Spec}(A)$ be an affine R-group. Its comultiplication μ, counit ϵ, and coinverse ι fit into the following commutative diagrams:

(a) coassociativity diagram

$$\begin{array}{ccc} A \otimes_R A \otimes_R A & \xleftarrow{\mu\otimes \mathrm{id}} & A \otimes_R A \\ \uparrow{\scriptstyle \mathrm{id}\otimes\mu} & & \uparrow{\scriptstyle \mu} \\ A \otimes_R A & \xleftarrow{\mu} & A \end{array}$$

(b) coidentity diagrams

$$\begin{array}{ccc} A \otimes_R R & \xleftarrow{\sim} & A \\ & {\scriptstyle \mathrm{id}\otimes\epsilon}\nwarrow \quad \swarrow{\scriptstyle \mu} & \\ & A \otimes_R A & \end{array} \qquad \begin{array}{ccc} R \otimes_R A & \xleftarrow{\sim} & A \\ & {\scriptstyle \epsilon\otimes\mathrm{id}}\nwarrow \quad \swarrow{\scriptstyle \mu} & \\ & A \otimes_R A & \end{array}$$

(c) coinverse diagram

$$\begin{array}{ccc} A & \underset{\mathrm{id}\otimes\iota}{\overset{\iota\otimes\mathrm{id}}{\leftleftarrows}} & A \otimes_R A \\ \uparrow & & \uparrow{\scriptstyle \mu} \\ R & \xleftarrow{\epsilon} & A \end{array}$$

PROOF. The assertion is evident by definition. □

Definition 1.1.9. Given an affine R-group $G = \operatorname{Spec}(A)$, we define its *augmentation ideal* to be the kernel of its counit $\epsilon : A \to R$.

LEMMA 1.1.10. For an affine R-group $G = \operatorname{Spec}(A)$ with augmentation ideal I, there exists a canonical R-module isomorphism $A \cong R \oplus I$.

PROOF. The assertion follows from the observation that the structure morphism $R \to A$ splits the short exact sequence

$$0 \longrightarrow I \longrightarrow A \xrightarrow{\epsilon} R \longrightarrow 0$$

where ϵ denotes the counit of G. □

PROPOSITION 1.1.11. Let G be an affine R-group.

(1) The unit section of G is a closed embedding.
(2) The kernel of an R-group homomorphism $f : H \to G$ is a closed R-subgroup of H.

PROOF. Let us write $G = \operatorname{Spec}(A)$ and denote its augmentation ideal by I. The first statement is evident as we naturally identify the unit section e of G with the closed embedding $\operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec}(A)$. The second statement follows from the first statement after identifying $\ker(f)$ with the fiber of f over e as noted in Lemma 1.1.5. □

Remark. Proposition 1.1.11 may fail for an R-group G which is not affine. In fact, the unit section of G is a closed embedding if and only if G is separated over R.

Example 1.1.12. Given an affine R-group G, its *n-torsion subgroup* $G[n] := \ker([n]_G)$ for each integer n is a closed R-subgroup of G by Proposition 1.1.11.

Remark. We have a natural identification $\mu_n \cong \mathbb{G}_m[n]$ for each integer $n \geq 1$.

Let us now introduce the objects of main interest for this section. For the rest of this section, we assume that R is noetherian unless stated otherwise.

Definition 1.1.13. Let $G = \mathrm{Spec}\,(A)$ be an affine group scheme over R.

(1) We say that G is *commutative* if it yields the commutative diagram

$$\begin{array}{ccc} G \times_R G & \xrightarrow{(g,h)\mapsto(h,g)} & G \times_R G \\ & {\scriptstyle m}\searrow \quad \swarrow{\scriptstyle m} & \\ & G & \end{array}$$

where m denotes the multiplication of G.

(2) We say that G is *finite flat of order* n if it is commutative with A being locally free of rank n over R.

Lemma 1.1.14. Let $G = \mathrm{Spec}\,(A)$ be an affine group scheme over R.

(1) G is commutative if and only if $G(B)$ is commutative for each R-algebra B.

(2) G is finite flat if and only if it is commutative with its structure morphism to $\mathrm{Spec}\,(R)$ being finite flat.

Proof. The first assertion is an immediate consequence of Lemma 1.1.2. The second assertion follows from a general fact stated in the Stacks Project [**Sta**, Tag 02KB]. ☐

Example 1.1.15. Some group schemes introduced in Example 1.1.7 are finite flat, as easily seen by their affine descriptions.

(1) The n-th roots of unity μ_n is finite flat of order n.

(2) If R is has characteristic p, the R-group α_p is finite flat of order p.

(3) For an abelian group M of order n, the constant R-group $\underline{M}$ is finite flat of order n.

Proposition 1.1.16. For an abelian scheme $\mathcal{A}$ of dimension g over R, its n-torsion subgroup $\mathcal{A}[n] = \ker([n]_{\mathcal{A}})$ is a finite flat R-group of order n^{2g}.

Proof. Since all fibers of $\mathcal{A}$ are abelian varieties of dimension g, the assertion follows from a standard fact about abelian varieties stated in the Stacks Project [**Sta**, Tag 03RP]. ☐

Many basic properties of finite abelian groups extend to finite flat group schemes. Here we state two fundamental theorems without a proof.

Theorem 1.1.17 (Deligne). Given a finite flat R-group G of order n, the homomorphism $[n]_G$ annihilates G in the sense that it factors through the unit section of G.

Remark. Curious reader can find Deligne's proof of Theorem 1.1.17 in the lecture notes of Stix [**Sti**, §3.3]. It is unknown whether Theorem 1.1.17 holds without the commutativity assumption on G.

Theorem 1.1.18 (Grothendieck [**Gro60**]). Let G be a finite flat R-group of order n with a finite flat closed R-subgroup H of order m.

(1) There exists a unique R-group G/H which fits into a short exact sequence

$$\underline{0} \longrightarrow H \longrightarrow G \longrightarrow G/H \longrightarrow \underline{0}.$$

(2) The R-group G/H is finite flat of order n/m.

Definition 1.1.19. Given a finite flat R-group G with a finite flat closed R-subgroup H, we refer to the R-group G/H in Theorem 1.1.18 as the *quotient group scheme of* G *by* H.

1.2. Cartier duality

In this subsection, we discuss a duality for finite flat R-groups. Given an R-module M, we write $M^\vee$ for its dual module. For an R-linear map f, we denote its dual map by $f^\vee$.

LEMMA 1.2.1. Let B be an R-algebra.

(1) Given an R-group G, the B-scheme G_B is naturally a B-group.
(2) Given a finite flat R-group G of order n, the B-group G_B is finite flat of order n.
(3) Given a short exact sequence of finite flat R-groups

$$\underline{0} \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow \underline{0},$$

the base change to B yields a short exact sequence

$$\underline{0} \longrightarrow (G')_B \longrightarrow G_B \longrightarrow (G'')_B \longrightarrow \underline{0}.$$

PROOF. The assertions are straightforward to verify by Lemma 1.1.2, Lemma 1.1.14, and a standard fact about finite flat morphisms stated in the Stacks project [**Sta**, Tag 02KD]. □

Definition 1.2.2. Given a finite flat R-group G, its *Cartier dual* $G^\vee$ is the group-valued functor on the category of R-algebras with

$$G^\vee(B) = \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B) \quad \text{for each } R\text{-algebra } B$$

where the group structure comes from the multiplication map on $(\mathbb{G}_m)_B$.

LEMMA 1.2.3. Given a finite flat R-group G with $[n]_G = 0$, we have

$$G^\vee(B) \cong \mathrm{Hom}_{B\text{-grp}}(G_B, (\mu_n)_B) \quad \text{for each } R\text{-algebra } B.$$

PROOF. The assertion follows immediately from the identification $\mu_n = \mathbb{G}_m[n]$. □

THEOREM 1.2.4 (Cartier duality). Let $G = \mathrm{Spec}\,(A)$ be a finite flat R-group of order n with comultiplication μ, counit ϵ, and coinverse ι. For the R-algebra A, we write $s : R \to A$ for its structure morphism and $m_A : A \otimes_R A \to A$ for its ring multiplication map.

(1) $A^\vee$ is an R-algebra with structure morphism $\epsilon^\vee$ and ring multiplication map $\mu^\vee$.
(2) $G^\vee$ is an R-group which admits a natural identification $G^\vee \cong \mathrm{Spec}\,(A^\vee)$ with comultiplication $m_A^\vee$, counit $s^\vee$, and coinverse $\iota^\vee$.
(3) $G^\vee$ is finite flat of order n.
(4) There exists a canonical R-group isomorphism $G \cong (G^\vee)^\vee$.

PROOF. Let us consider the natural identifications

$$R^\vee \cong R \quad \text{and} \quad (A \otimes_R A)^\vee \cong A^\vee \otimes_R A^\vee.$$

The map $\mu^\vee$ fits into associativity and commutativity diagrams induced by the corresponding diagrams for the multiplication on G. In addition, we have commutative diagrams

$$\begin{array}{ccc} A^\vee \otimes_R R & \xrightarrow{\sim} & A^\vee \\ & {\scriptstyle \mathrm{id}\otimes\epsilon^\vee} \searrow \quad \nearrow {\scriptstyle \mu^\vee} & \\ & A^\vee \otimes_R A^\vee & \end{array} \qquad \begin{array}{ccc} R \otimes_R A^\vee & \xrightarrow{\sim} & A^\vee \\ & {\scriptstyle \epsilon^\vee\otimes\mathrm{id}} \searrow \quad \nearrow {\scriptstyle \mu^\vee} & \\ & A^\vee \otimes_R A^\vee & \end{array}$$

induced by the identity diagrams for G. Hence we deduce statement (1).

Let us now consider statement (2). It is straightforward to verify that $G^{\triangledown} := \mathrm{Spec}\,(A^\vee)$ is an R-group with comultiplication $m_A^\vee$, counit $s^\vee$, and coinverse $\iota^\vee$. Let B be an arbitrary R-algebra. In light of Lemma 1.1.2, we wish to establish a canonical isomorphism

$$G^\vee(B) \cong G^{\triangledown}(B). \tag{1.1}$$

Let μ_B, ϵ_B, and ι_B respectively denote the comultiplication, counit, and coinverse of $G_B \cong \mathrm{Spec}\,(A_B)$. By the affine description of $\mathbb{G}_m$ given in Example 1.1.7, we find

$$G^\vee(B) = \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B) \cong \left\{\, f \in \mathrm{Hom}_{B\text{-alg}}(B[t,t^{-1}], A_B) : \mu_B(f(t)) = f(t) \otimes f(t) \,\right\}$$

where the identity $\mu_B(f(t)) = f(t) \otimes f(t)$ comes from compatibility with comultiplications. Since we have the canonical isomorphism $\mathrm{Hom}_{B\text{-alg}}(B[t,t^{-1}], A_B) \cong A_B^\times$ which sends each $f \in \mathrm{Hom}_{B\text{-alg}}(B[t,t^{-1}], A_B)$ to $f(t)$, we obtain a natural identification

$$G^\vee(B) \cong \left\{\, u \in A_B^\times : \mu_B(u) = u \otimes u \,\right\}. \tag{1.2}$$

Meanwhile, as $A_B^\vee$ is a B-algebra by statement (1), we have

$$G^{\triangledown}(B) \cong \mathrm{Hom}_{R\text{-alg}}(A^\vee, B) \cong \mathrm{Hom}_{B\text{-alg}}(A_B^\vee, B). \tag{1.3}$$

Let us denote the ring multiplication map on B by m_B and the identity map on B by id_B. By definition, $\mathrm{Hom}_{B\text{-alg}}(A_B^\vee, B)$ is the group of B-module homomorphisms $A_B^\vee \to B$ through which $\mu_B^\vee$ and $\epsilon_B^\vee$ are respectively compatible with m_B and id_B. Taking B-duals, we identify this group with the group of B-module homomorphisms $B \to A_B$ through which $m_B^\vee$ and $\mathrm{id}_B^\vee$ are respectively compatible with μ_B and ϵ_B. Since we have the canonical isomorphism $\mathrm{Hom}_{B\text{-alg}}(B, A_B) \cong A_B^\times$ which sends each $f \in \mathrm{Hom}_{B\text{-alg}}(B, A_B)$ to $f(1)$, we find

$$\mathrm{Hom}_{B\text{-alg}}(A_B^\vee, B) \cong \left\{\, u \in A_B^\times : \mu_B(u) = u \otimes u,\ \epsilon_B(u) = 1 \,\right\}. \tag{1.4}$$

Moreover, the group scheme axioms for G_B yields the relation $(\epsilon_B \otimes \mathrm{id}_B) \circ \mu_B = \mathrm{id}_B$ and consequently implies that every $u \in A_B^\times$ with $\mu_B(u) = u \otimes u$ must satisfy the identity $\epsilon_B(u) = 1$. Hence the isomorphisms (1.3) and (1.4) together yield a natural identification

$$G^{\triangledown}(B) \cong \left\{\, u \in A_B^\times : \mu_B(u) = u \otimes u \,\right\}. \tag{1.5}$$

Now we establish the desired isomorphism (1.1) by the identifications (1.2) and (1.5), thereby completing the proof of statement (2).

It remains to prove statements (3) and (4). Since $G^\vee$ is commutative by Lemma 1.1.14 and the commutativity of $\mathbb{G}_m$, we deduce statement (3) from statement (2) by observing that $A^\vee$ is locally free of rank n over R. In addition, we apply statements (1) and (2) to see that the canonical R-module isomorphism $A \cong (A^\vee)^\vee$ is indeed an R-algebra isomorphism which respects comultiplications, counits, and coinverses on both sides, thereby establishing statement (4). ☐

Proposition 1.2.5. Given a finite flat R-group G and an R-algebra B, there exists a natural B-group isomorphism $G^\vee \times_R B \cong (G \times_R B)^\vee$.

Proof. It is evident that $G^\vee \times_R B$ and $(G \times_R B)^\vee$ are naturally isomorphic as group-valued functors. Lemma 1.2.1 and Theorem 1.2.4 together imply that these functors are indeed finite flat B-groups and thus yield the desired assertion. ☐

Definition 1.2.6. Given a homomorphism $f : G \to H$ of finite flat R-groups, we refer to the induced homomorphism $f^\vee : G^\vee \to H^\vee$ as the *dual homomorphism* of f.

Example 1.2.7. Given a finite flat R-group G, we have $[n]_G^\vee = [n]_{G^\vee}$ for every integer $n > 0$; indeed, $[n]_G^\vee$ maps each $f \in G^\vee(B) = \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B)$ for an arbitrary R-algebra B to $f \circ [n]_{G_B} = [n]_{G^\vee}(f)$.

PROPOSITION 1.2.8. For every positive integer n, we have $(\underline{\mathbb{Z}/n\mathbb{Z}})^\vee \cong \mu_n$ and $\mu_n^\vee \cong \underline{\mathbb{Z}/n\mathbb{Z}}$.

PROOF. Let us set $A := \prod_{i \in \mathbb{Z}/n\mathbb{Z}} R$ and write e_i for the element of A whose only nonzero entry is 1 in the component corresponding to i. As explained in Example 1.1.7 we have $\underline{\mathbb{Z}/n\mathbb{Z}} \cong \operatorname{Spec}(A)$ with comultiplication μ, counit ϵ, and coinverse ι given by the relations

$$\mu(e_i) = \sum_{v+w=i} e_v \otimes e_w, \quad \epsilon(e_i) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \iota(e_i) = e_{-i}.$$

Let $m_A : A \otimes_R A \to A$ and $s : R \to A$ respectively denote the ring multiplication map and structure morphism of A. We have the dual basis (f_i) of $A^\vee$ with

$$f_i(e_j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.2.4 yields a natural identification $(\underline{\mathbb{Z}/n\mathbb{Z}})^\vee \cong \operatorname{Spec}(A^\vee)$ with comultiplication $m_A^\vee$, counit $s^\vee$, and coinverse $\iota^\vee$, where $A^\vee$ is an R-algebra with structure morphism $\epsilon^\vee$ and ring multiplication map $\mu^\vee$. The maps $\mu^\vee$, $\epsilon^\vee$, $m_A^\vee$, $s^\vee$, and $\iota^\vee$ are determined by the identities

$$\mu^\vee(f_i \otimes f_j) = f_{i+j}, \quad \epsilon^\vee(1) = f_0, \quad m_A^\vee(f_i) = f_i \otimes f_i, \quad s^\vee(f_i) = 1, \quad \iota^\vee(f_i) = f_{-i}.$$

Hence the map $A^\vee \to R[t]/(t^n - 1)$ sending each f_i to t^i induces an R-group isomorphism $(\underline{\mathbb{Z}/n\mathbb{Z}})^\vee \cong \mu_n$ by Example 1.1.7 and in turn yields an R-group isomorphism $\mu_n^\vee \cong \underline{\mathbb{Z}/n\mathbb{Z}}$ by Theorem 1.2.4. □

PROPOSITION 1.2.9. If R has characteristic p, the R-group α_p is self-dual.

PROOF. As explained in Example 1.1.7, we have $\alpha_p = \operatorname{Spec}(A)$ for $A := R[t]/(t^p)$ with comultiplication μ, counit ϵ, and coinverse ι given by the relations

$$\mu(t^i) = \sum_{v+w=i} \binom{i}{v} t^v \otimes t^w, \quad \epsilon(t^i) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \iota(t^i) = (-t)^i.$$

Let $m_A : A \otimes_R A \to A$ and $s : R \to A$ respectively denote the ring multiplication map and structure morphism of A. We have the dual basis (f_i) of $A^\vee$ with

$$f_i(t^j) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.2.4 yields a canonical identification $\alpha_p^\vee \cong \operatorname{Spec}(A^\vee)$ with comultiplication $m_A^\vee$, counit $s^\vee$, and coinverse $\iota^\vee$, where $A^\vee$ is an R-algebra with structure morphism $\epsilon^\vee$ and ring multiplication map $\mu^\vee$. The maps $\mu^\vee$, $\epsilon^\vee$, $m_A^\vee$, $s^\vee$, and $\iota^\vee$ are determined by the identities

$$\mu^\vee(f_i \otimes f_j) = \binom{i+j}{i} f_{i+j}, \quad \epsilon^\vee(1) = 0,$$

$$m_A^\vee(f_i) = \sum_{v+w=i} f_v \otimes f_w, \quad s^\vee(f_i) = \begin{cases} 1 & \text{for } i = 0 \\ 0 & \text{otherwise} \end{cases}, \quad \iota^\vee(f_i) = (-1)^i f_i.$$

Hence the map $A^\vee \to A$ sending each f_i to $t^i/i!$ yields an R-group isomorphism $\alpha_p^\vee \cong \alpha_p$. □

Remark. When R has characteristic p, we have an R-scheme isomorphism $\mu_p \simeq \alpha_p$ given by the ring isomorphism $R[t]/(t^p) \simeq R[t]/(t^p - 1)$ sending t to $t+1$. Propositions 1.2.8 and 1.2.9 together show that μ_p and α_p are not isomorphic as group schemes.

PROPOSITION 1.2.10. Given an abelian scheme $\mathcal{A}$ over R with dual abelian scheme $\mathcal{A}^\vee$, we have a natural isomorphism $\mathcal{A}[n]^\vee \cong \mathcal{A}^\vee[n]$ for every positive integer n.

PROOF. The homomorphism $[n]_{\mathcal{A}}$ is surjective by a standard fact about abelian varieties stated in the Stacks Project [**Sta**, Tag 03RP]. Hence we have a short exact sequence

$$\underline{0} \longrightarrow \mathcal{A}[n] \longrightarrow \mathcal{A} \xrightarrow{[n]} \mathcal{A} \longrightarrow \underline{0}$$

which gives rise to a long exact sequence

$$\underline{0} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{A}, \mathbb{G}_m) \xrightarrow{[n]} \underline{\mathrm{Hom}}(\mathcal{A}, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{A}[n], \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1(\mathcal{A}, \mathbb{G}_m) \xrightarrow{[n]} \underline{\mathrm{Ext}}^1(\mathcal{A}, \mathbb{G}_m).$$

In addition, we have natural identifications

$$\underline{\mathrm{Hom}}(\mathcal{A}, \mathbb{G}_m) = \underline{0}, \quad \underline{\mathrm{Hom}}(\mathcal{A}[n], \mathbb{G}_m) \cong \mathcal{A}[n]^\vee, \quad \underline{\mathrm{Ext}}^1(\mathcal{A}, \mathbb{G}_m) \cong \mathcal{A}^\vee$$

by definition of Cartier duals and some general fact about abelian varieties stated in the notes of Milne [**Mil**, §9]. Therefore we obtain an exact sequence

$$\underline{0} \longrightarrow \mathcal{A}[n]^\vee \longrightarrow \mathcal{A}^\vee \xrightarrow{[n]} \mathcal{A}^\vee$$

which yields the desired isomorphism $\mathcal{A}[n]^\vee \cong \mathcal{A}^\vee[n]$. □

Example 1.2.11. If $R = k$ is a field, every elliptic curve E over k admits a natural isomorphism $E[n]^\vee \cong E[n]$ for each integer $n \geq 1$ by Proposition 1.2.10 and a standard fact that elliptic curves are self-dual as stated in the notes of Milne [**Mil**, §9].

LEMMA 1.2.12. Given a closed embedding $f : H \hookrightarrow G$ of finite flat R-groups, there exists a canonical isomorphism $\ker(f^\vee) \cong (G/H)^\vee$.

PROOF. Let B be an arbitrary R-algebra and $f_B : H_B \hookrightarrow G_B$ denote the homomorphism induced by f. Theorem 1.1.18 and Lemma 1.2.1 together yield a canonical isomorphism $G_B/H_B \cong (G/H)_B$. Hence we obtain an identification

$$\begin{aligned} \ker(f^\vee)(B) &= \{\, g \in \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B) : g \circ f_B = 0 \,\} \\ &= \{\, g \in \mathrm{Hom}_{B\text{-grp}}(G_B, (\mathbb{G}_m)_B) : H_B \subseteq \ker(g) \,\} \\ &\cong \mathrm{Hom}_{B\text{-grp}}(G_B/H_B, (\mathbb{G}_m)_B) \cong \mathrm{Hom}_{B\text{-grp}}((G/H)_B, (\mathbb{G}_m)_B) = (G/H)^\vee(B), \end{aligned}$$

thereby establishing the desired assertion. □

PROPOSITION 1.2.13. Given a short exact sequence of finite flat R-groups

$$\underline{0} \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow \underline{0},$$

the Cartier duality gives rise to a short exact sequence

$$\underline{0} \longrightarrow G''^\vee \longrightarrow G^\vee \longrightarrow G'^\vee \longrightarrow \underline{0}.$$

PROOF. Let f and g respectively denote the maps $G' \to G$ and $G \to G''$ in the given short exact sequence. It is straightforward to verify the injectivity of $g^\vee$ by the surjectivity of g. In addition, Lemma 1.2.12 yields a canonical isomorphism $\ker(f^\vee) \cong G''^\vee$. Therefore it remains to establish the surjectivity of $f^\vee$. Since $G''^\vee$ is a finite flat closed R-subgroup of $G^\vee$ by Proposition 1.1.11 and Theorem 1.2.4, we obtain the quotient R-group $G^\vee/G''^\vee$ by Theorem 1.1.18. Now $f^\vee$ factors through a homomorphism $G^\vee/G''^\vee \to G'^\vee$, whose dual coincides with the isomorphism $\ker(g) \cong G'$ induced by f under the identifications

$$(G'^\vee)^\vee \cong G' \quad \text{and} \quad (G^\vee/G''^\vee)^\vee \cong \ker((g^\vee)^\vee) \cong \ker(g)$$

given by Theorem 1.2.4 and Lemma 1.2.12. Hence we deduce that $f^\vee$ is surjective as desired, thereby completing the proof. □

1.3. Finite étale group schemes

In this subsection, we introduce finite étale group schemes and discuss their properties.

Definition 1.3.1. We say that an affine R-group $G = \mathrm{Spec}\,(A)$ is *finite étale* if it is finite flat with $\Omega_{A/R} = 0$, where $\Omega_{A/R}$ denotes the module of relative differentials.

LEMMA 1.3.2. Let $G = \mathrm{Spec}\,(A)$ be a commutative affine R-group.

(1) G is finite étale if and only if its structure morphism to $\mathrm{Spec}\,(R)$ is finite étale.
(2) When $R = k$ is a field, G is finite étale if and only if there exists a k-algebra isomorphism $A \simeq \prod_{i=1}^{n} k_i$ where each k_i is a finite separable extension of k.

PROOF. The first assertion is an immediate consequence of Lemma 1.1.14. The second assertion follows from the first assertion by a standard fact about étale morphisms stated in the Stacks project [**Sta**, Tag 00U3]. □

LEMMA 1.3.3. Given a finite étale R-group G and an R-algebra B, the B-scheme G_B is a finite étale B-group.

PROOF. The assertion follows from Lemma 1.2.1, Lemma 1.3.2, and a standard fact that a base change of an étale morphism is étale as stated in the Stacks project [**Sta**, Tag 02GO]. □

PROPOSITION 1.3.4. Assume that R is a henselian local ring with perfect residue field k.

(1) There exists an equivalence of categories
$$\{\text{ finite étale } R\text{-groups }\} \xrightarrow{\sim} \{\text{ finite abelian groups with a continuous } \Gamma_k\text{-action }\}$$
which sends each finite étale R-group G to $G(\overline{k})$.
(2) If a finite étale R-group G has order n, the abelian group $G(\overline{k})$ also has order n.

PROOF. Let us first consider statement (1). By some standard facts about finite étale morphisms stated in the Stacks project [**Sta**, Tag 09ZS and Tag 0BQ8], there exists an equivalence of categories
$$\{\text{ finite étale } R\text{-schemes }\} \xrightarrow{\sim} \{\text{ finite sets with a continuous } \Gamma_k\text{-action }\}$$
which maps each R-scheme T to $T(\overline{k})$. Hence we obtain the desired equivalence by passing to the corresponding categories of commutative group objects.

For statement (2), we write $G = \mathrm{Spec}\,(A)$ for some locally free R-algebra A of rank n. By Lemma 1.3.2 and Lemma 1.3.3, there exists a k-algebra isomorphism $A \otimes_R k \simeq \prod_{i=1}^{m} k_i$ where each k_i is a finite separable extension of k. Hence we find
$$G(\overline{k}) \cong \mathrm{Hom}_{R\text{-alg}}(A, \overline{k}) \cong \mathrm{Hom}_{R\text{-alg}}(A \otimes_R k, \overline{k}) \simeq \mathrm{Hom}_{R\text{-alg}}(\prod_{i=1}^{m} k_i, \overline{k}) \cong \prod_{i=1}^{m} \mathrm{Hom}_k(k_i, \overline{k})$$
and in turn deduce that the order of $G(\overline{k})$ is
$$\sum_{i=1}^{m} \dim_k(k_i) = \dim_k(A \otimes_R k) = n,$$
thereby completing the proof. □

Remark. Primary examples of henselian local rings are complete local rings and fields.

Proposition 1.3.5. Let $G = \mathrm{Spec}\,(A)$ be a finite flat R-group with augmentation ideal I.

(1) There exist natural isomorphisms

$$I/I^2 \otimes_R A \cong \Omega_{A/R} \quad \text{and} \quad I/I^2 \cong \Omega_{A/R} \otimes_A A/I.$$

(2) G is étale if and only if we have $I = I^2$.

Proof. Let us consider a commutative diagram

$$\begin{array}{ccc} G \times_R G & \xrightarrow{(g,h)\mapsto(g,gh^{-1})} & G \times_R G \\ & {\scriptstyle\Delta}\nwarrow \quad \nearrow{\scriptstyle(\mathrm{id},e)} & \\ & G & \end{array}$$

where Δ and e respectively denote the diagonal morphism and the unit section of G. The horizontal map is an R-scheme isomorphism whose inverse sends each $(g, h) \in G \times_R G$ to $(g, h^{-1}g) \in G \times_R G$. Hence we obtain a commutative diagram

$$\begin{array}{ccc} A \otimes_R A & \xleftarrow{\sim} & A \otimes_R A \\ & {\scriptstyle a\otimes b\mapsto ab}\searrow \quad \swarrow{\scriptstyle a\otimes b\mapsto a\cdot\epsilon(b)} & \\ & A & \end{array}$$

where ϵ denotes the counit of G. The horizontal map induces an isomorphism between the kernels of the two downward maps. Let J denote the kernel of the left downward map. Under the canonical decomposition

$$A \otimes_R A \cong A \otimes_R R \oplus A \otimes_R I$$

given by Lemma 1.1.10, we identify the kernel of the right downward map with $A \otimes_R I$ and consequently obtain a natural isomorphism $J \cong A \otimes_R I$. Now we have

$$\Omega_{A/R} \cong J/J^2 \cong (A \otimes_R I)/(A \otimes_R I)^2 \cong (A \otimes_R I)/(A \otimes_R I^2) \cong A \otimes_R (I/I^2),$$

where the first identification comes from a standard fact about relative differentials stated in the Stacks project [**Sta**, Tag 00RW], and thus find

$$\Omega_{A/R} \otimes_A (A/I) \cong \big((I/I^2) \otimes_R A\big) \otimes_A A/I \cong (I/I^2) \otimes_R A/I \cong (I/I^2) \otimes_R R \cong I/I^2,$$

thereby establishing statement (1). Statement (2) immediately follows from statement (1). □

Remark. Let us sketch a slightly different proof of Proposition 1.3.5, which provides some geometric intuition behind our argument. The R-group G induces a natural action on the R-module $\Omega_{G/\mathrm{Spec}\,(R)} = \Omega_{A/R}$ by translations. Let $\omega_{A/R}$ denote the R-module of invariant elements in $\Omega_{A/R}$ under this action. It is not hard to show by adapting our argument that there exist canonical isomorphisms

$$\omega_{A/R} \otimes_R A \cong \Omega_{A/R} \quad \text{and} \quad \omega_{A/R} \cong \Omega_{A/R} \otimes_A A/I.$$

The first isomorphism says that we can get every element in $\Omega_{A/R}$ by multiplying a global section on G to an invariant element, while the second isomorphism implies that we can determine every invariant element in $\Omega_{A/R}$ by its pullback along the unit section. Meanwhile, since we have the conormal exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{A/R} \otimes_A A/I \longrightarrow \Omega_{R/R} \longrightarrow 0$$

given by a standard fact stated in the Stacks project [**Sta**, Tag 06AA], we obtain the identification $I/I^2 \cong \Omega_{A/R} \otimes_A A/I$ by observing that $\Omega_{R/R}$ vanishes. Now we find $\omega_{A/R} \cong I/I^2$ and consequently establish the desired assertions.

Proposition 1.3.6. Every finite flat constant group scheme is étale.

Proof. Let M be a finite abelian group with identity element denoted by 0. We denote by I the augment ideal of $\underline{M}$. By the affine description in Example 1.1.7, we have

$$\underline{M} \cong \operatorname{Spec}\left(\prod_{i \in M} R\right) \quad \text{and} \quad I = \prod_{\substack{i \in M \\ i \neq 0}} R.$$

Hence we obtain the identity $I = I^2$ and in turn deduce from Proposition 1.3.5 that $\underline{M}$ is étale as desired. □

Proposition 1.3.7. Assume that $R = k$ is an algebraically closed field.

(1) Every finite étale k-groups is a constant group scheme.
(2) Given a prime p, the k-group $\underline{\mathbb{Z}/p\mathbb{Z}}$ is a unique finite étale k-group of order p.

Proof. Proposition 1.3.4 yields an equivalence of categories

$$\{ \text{ finite étale } k\text{-groups } \} \xrightarrow{\sim} \{ \text{ finite abelian groups } \}$$

which sends each finite étale k-group G to $G(k)$. Meanwhile, for every finite abelian group M, the constant group scheme $\underline{M}$ admits a natural isomorphism $\underline{M}(k) \cong M$ by Example 1.1.7. Hence we establish the desired assertions by Proposition 1.3.6 and the fact that $\mathbb{Z}/p\mathbb{Z}$ is a unique group of order p. □

Proposition 1.3.8. A finite flat R-group G is étale if and only if the (scheme theoretic) image of the unit section is open.

Proof. We write $G = \operatorname{Spec}(A)$ for some locally free R-algebra A of finite rank. In addition, we denote by I the augmentation ideal of G. The (scheme theoretic) image of the unit section is the closed subscheme $\operatorname{Spec}(A/I)$ of $\operatorname{Spec}(A)$.

Let us first assume that G is étale. Proposition 1.3.5 shows that I/I^2 vanishes. Hence by Nakayama's lemma there exists an element $a \in A$ with $a - 1 \in I$ and $aI = 0$. We obtain the equality $a^2 = a(a-1) + a = a$, which means that a is idempotent. Let us consider the localization map $A \to A_a$, which is surjective as we have

$$\frac{b}{a^n} = \frac{ba}{a^{n+1}} = \frac{ba}{a} = \frac{b}{1} \quad \text{for each } b \in A \text{ and } n \geq 1.$$

Its kernel consists of elements $b \in A$ with $a^n b = 0$ for some $n \geq 1$, or equivalently $ab = 0$. We see that the kernel contains I by the identity $aI = 0$, while every element b in the kernel satisfies the relation

$$b = -(a-1)b + ab = -(a-1)b \in I.$$

Hence the localization map $A \to A_a$ has I as its kernel and yields an isomorphism $A/I \cong A_a$. We deduce that the closed embedding $\operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec}(A)$ is open.

For the converse, we now assume that the embedding $\operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec}(A)$ is open. Since open embeddings are flat as stated in the Stacks project [**Sta**, Tag 0250], the ring homomorphism $A \twoheadrightarrow A/I$ must be flat. Therefore we obtain a short exact sequence

$$0 \longrightarrow I \otimes_A A/I \longrightarrow A \otimes_A A/I \longrightarrow A/I \otimes_A A/I \longrightarrow 0,$$

which in turn yields a short exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow A/I \longrightarrow A/I \longrightarrow 0$$

with the third arrow being the identity map. We see that I/I^2 vanishes and in turn deduce from Proposition 1.3.5 that G is étale as desired. □

THEOREM 1.3.9. A finite flat R-group G with order invertible in R must be étale.

PROOF. Let us write $G = \mathrm{Spec}\,(A)$ for some locally free R-algebra A of finite rank. The group axioms for G yield commutative diagrams

$$\begin{array}{ccc} \mathrm{Spec}\,(R) & \xrightarrow{\;e\;} & G \\ {\scriptstyle (e,e)}\downarrow & \nearrow_{m} & \\ G \times_R G & & \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\;\mathrm{id}\;} & G \\ {\scriptstyle (\mathrm{id},e)}\downarrow\downarrow{\scriptstyle (e,\mathrm{id})} & \nearrow_{m} & \\ G \times_R G & & \end{array}$$

where m and e respectively denote the multiplication map and unit section of G. These diagrams are equivalent to the commutative diagrams

$$\begin{array}{ccc} R & \xleftarrow{\;\epsilon\;} & A \\ {\scriptstyle \epsilon\otimes\epsilon}\uparrow & \swarrow_{\mu} & \\ A \otimes_R A & & \end{array} \qquad \begin{array}{ccc} A & \xleftarrow{\;\mathrm{id}\;} & A \\ {\scriptstyle \mathrm{id}\otimes\epsilon}\uparrow\uparrow{\scriptstyle \epsilon\otimes\mathrm{id}} & \swarrow_{\mu} & \\ A \otimes_R A & & \end{array} \tag{1.6}$$

where μ and ϵ respectively denotes the comultiplication and counit of G. Let us denote the augmentation ideal of G by I and take an arbitrary element $t \in I$. We have $\mu(t) \in \ker(\epsilon \otimes \epsilon)$ by the diagram (1.6). Moreover, under the decomposition

$$A \otimes_R A \cong (R \otimes_R R) \oplus (I \otimes_R R) \oplus (R \otimes_R I) \oplus (I \otimes_R I)$$

given by Lemma 1.1.10, we obtain a natural identification

$$\ker(\epsilon \otimes \epsilon) \cong (I \otimes_R R) \oplus (R \otimes_R I) \oplus (I \otimes_R I)$$

and thus find $\mu(t) \in a \otimes 1 + 1 \otimes b + I \otimes_R I$ for some $a, b \in I$. Now the diagram (1.6) implies that a and b are both equal to t, thereby yielding the relation

$$\mu(t) \in t \otimes 1 + 1 \otimes t + I \otimes_R I. \tag{1.7}$$

We assert that $[n]_G$ for each $n \geq 1$ induces multiplication by n on I/I^2. Let $[n]_A : A \to A$ denote the R-algebra homomorphism induced by $[n]_G$. We have commutative diagrams

$$\begin{array}{ccc} G & \xrightarrow{\;[n]_G\;} & G \\ {\scriptstyle ([n-1]_G,\mathrm{id})}\downarrow & \nearrow_{m} & \\ G \times_R G & & \end{array} \qquad \begin{array}{ccc} A & \xleftarrow{\;[n]_A\;} & A \\ {\scriptstyle [n-1]_A\otimes\mathrm{id}}\uparrow & \swarrow_{\mu} & \\ A \otimes_R A & & \end{array}$$

and thus apply the relation (1.7) to find $[n]_A(t) \in [n-1]_A(t) + t + I^2$. Since $[1]_A$ is the identity map on A, a simple induction yields the relation $[n]_A(t) \in nt + I^2$ for each $n \geq 1$. Hence we obtain the desired assertion as t is an arbitrary element in I.

Let us denote the order of G by m. Theorem 1.1.17 shows that $[m]_G$ factors through the unit section of G, which implies that the induced map on $\Omega_{A/R}$ factors through $\Omega_{R/R} = 0$. We find that $[m]_G$ induces a zero map on $I/I^2 \cong \Omega_{A/R} \otimes_A A/I$ by Proposition 1.3.5. Meanwhile, $[m]_G$ induces the multiplication by m on I/I^2, which is an isomorphism as m is invertible in R. Hence we deduce that I/I^2 vanishes, thereby completing the proof by Proposition 1.3.5. $\square$

Remark. Theorem 1.3.9 is the only result which relies on Theorem 1.1.17 in our discussion. If R is a field, it is possible to prove Theorem 1.3.9 without using Theorem 1.1.17.

COROLLARY 1.3.10. Every finite flat group scheme over a field of characteristic 0 is étale.

1.4. The connected-étale sequence

Throughout this subsection, we assume that R is a henselian local ring and denote its residue field by k. Our main goal for this subsection is to discuss a fundamental theorem that every finite flat R-group naturally arises as an extension of an étale R-group by a connected R-group.

LEMMA 1.4.1. A finite flat R-scheme is étale if and only if its special fiber is étale.

PROOF. The assertion immediately follows from some standard facts about étale morphisms stated in the Stacks project [**Sta**, Tag 02GO, Tag 02GM, and Tag 00U3]. ☐

Remark. Our proof shows that Lemma 1.4.1 does not require R to be henselian.

LEMMA 1.4.2. A finite R-scheme T is connected if and only if it satisfies the following equivalent conditions:

(i) T is a spectrum of a henselian local finite R-algebra.
(ii) The action of Γ_k on $T(\overline{k})$ is transitive.

PROOF. Let us write $T = \mathrm{Spec}\,(B)$ for some finite R-algebra B. By a general fact about henselian local rings stated in the Stacks project [**Sta**, Tag 04GH], we have

$$B \simeq \prod_{i=1}^{n} B_i$$

where each B_i is a henselian local finite R-algebra. Since the spectrum of a local ring is connected, each $T_i := \mathrm{Spec}\,(B_i)$ corresponds to a connected component of T. Hence T is connected if and only if it satisfies condition (i).

We denote the residue field of each B_i by k_i. Via the isomorphism

$$T(\overline{k}) = \mathrm{Hom}_{R\text{-alg}}(B, \overline{k}) \simeq \coprod_{i=1}^{n} \mathrm{Hom}_k(k_i, \overline{k}),$$

we identify each $\mathrm{Hom}_k(k_i, \overline{k})$ as an orbit under the action of Γ_k on $T(\overline{k})$. Therefore T is connected if and only if it satisfies condition (ii). ☐

Remark. When k is algebraically closed, a finite R-scheme T is connected if and only if $T(k)$ is a singleton by Lemma 1.4.2.

LEMMA 1.4.3. A finite R-scheme is connected if and only if its special fiber is connected.

PROOF. The assertion is evident by Lemma 1.4.2. ☐

Remark. Lemma 1.4.3 is a special case of a general fact that for every proper R-scheme T there exists a natural bijection between the connected components of T and the connected components of T_k, as stated in SGA 4 1/2, Exp. 1, Proposition 4.2.1.

LEMMA 1.4.4. Connected components of a finite flat R-scheme T are finite flat over R.

PROOF. Let T° be a connected component of T. The closed embedding $T^\circ \hookrightarrow T$ is finite flat by general facts stated in the Stacks project [**Sta**, Tag 035C, Tag 04PX]. Hence T° is finite flat over R by a standard fact that the composition of finite flat morphisms is finite flat as stated in the Stacks project [**Sta**, Tag 01WK, Tag 01U7]. ☐

Remark. Our proof shows that Lemma 1.4.4 holds without any assumption on the base ring.

Definition 1.4.5. Given an R-group G, its *identity component* G° is the connected component of the unit section.

LEMMA 1.4.6. For a finite flat R-group G, we have $G^\circ(\overline{k}) = 0$.

PROOF. Let us write $G = \mathrm{Spec}\,(A)$ for some locally free R-algebra A of finite rank. By Lemma 1.4.2 and Lemma 1.4.4, we have $G^\circ = \mathrm{Spec}\,(A^\circ)$ for some henselian local finite R-algebra A°. Since the unit section factors through G°, it induces a surjective ring homomorphism $A^\circ \to R$. We denote its kernel by I° and obtain an isomorphism $A^\circ/I^\circ \cong R$, which induces an isomorphism between the residue fields of A° and R. Hence we find

$$G^\circ(\overline{k}) = \mathrm{Hom}_{R\text{-alg}}(A^\circ, \overline{k}) \cong \mathrm{Hom}_k(k, \overline{k}) = 0$$

as desired. □

PROPOSITION 1.4.7. A finite flat R-group G is connected if and only if we have $G(\overline{k}) = 0$.

PROOF. If $G(\overline{k})$ is trivial, G is connected by Lemma 1.4.2. Conversely, if G is connected, we have $G = G^\circ$ and thus find $G(\overline{k}) = 0$ by Lemma 1.4.6. □

Example 1.4.8. Let us present some primary examples of connected R-groups.

(1) If k has characteristic p, the R-group μ_{p^v} for each integer $v \geq 1$ is connected by Proposition 1.4.7.

(2) If R has characteristic p, the R-group α_p is connected by Proposition 1.4.7.

THEOREM 1.4.9. Let G be a finite flat R-group. The identity component G° is naturally a finite flat closed R-subgroup of G such that the quotient $G^{\text{ét}} := G/G^\circ$ is étale.

PROOF. Let us first prove that G° is a finite flat closed R-subgroup of G. Since we have $(G^\circ \times_R G^\circ)(\overline{k}) \cong G^\circ(\overline{k}) \times G^\circ(\overline{k}) = 0$ by Lemma 1.4.6, the scheme $G^\circ \times_R G^\circ$ is connected by Lemma 1.4.2. Hence the image of $G^\circ \times_R G^\circ$ under the multiplication map lies in G° for being a connected subscheme of G which contains the unit section. Similarly, the image of G° under the inverse map lies in G°. Therefore G° is an R-subgroup of G, which is evidently closed by construction. Moreover, G° is finite flat by Lemma 1.1.14 and Lemma 1.4.4.

We now consider the finite flat R-group $G^{\text{ét}} = G/G^\circ$ given by Theorem 1.1.18. Its unit section G°/G° has an open image as G° is open in G by the noetherian hypothesis on R. Hence we deduce from Proposition 1.3.8 that $G^{\text{ét}}$ is étale, thereby completing the proof. □

Definition 1.4.10. Given a finite flat R-group G, we refer to the short exact sequence

$$\underline{0} \longrightarrow G^\circ \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow \underline{0}$$

given by Theorem 1.4.9 as the *connected-étale sequence* of G.

Example 1.4.11. Let us describe the connected-étale sequence of μ_n for each integer $n \geq 1$. If k has characteristic 0, Corollary 1.3.10 and Lemma 1.4.1 together imply that μ_n is étale, thereby yielding the connected-étale sequence

$$\underline{0} \longrightarrow \underline{0} \longrightarrow \mu_n \xrightarrow{\text{id}} \mu_n \longrightarrow \underline{0}.$$

Let us henceforth assume that k has characteristic p. We may write $n = p^v m$ for some positive integers v and m such that m is not divisible by p. Then we have a short exact sequence

$$\underline{0} \longrightarrow \mu_{p^v} \longrightarrow \mu_n \xrightarrow{[p^v]} \mu_m \longrightarrow \underline{0}. \tag{1.8}$$

The R-group μ_{p^v} is connected as noted in Example 1.4.8. Moreover, since μ_m has order m by Example 1.1.15, it is étale as easily seen by Theorem 1.3.9 and Lemma 1.4.1. Hence the exact sequence (1.8) is indeed the connected-étale sequence of μ_n.

Proposition 1.4.12. Let G be a finite flat R-group.

(1) The natural surjection $G \twoheadrightarrow G^{\text{ét}}$ induces a canonical isomorphism $G(\overline{k}) \cong G^{\text{ét}}(\overline{k})$.
(2) G is étale if and only if we have $G^\circ = \underline{0}$.

Proof. The first statement is evident by Lemma 1.4.6 and Theorem 1.4.9. Since the (scheme theoretic) image of the unit section is closed as noted in Proposition 1.1.11, it is open if and only if it coincides with its connected component G°. Therefore the second statement follows from Proposition 1.3.8. □

Proposition 1.4.13. Let $f : G \to H$ be a homomorphism of finite flat R-groups.

(1) If G is connected, f factors through the embedding $H^\circ \hookrightarrow H$.
(2) If H is étale, f factors through the surjection $G \twoheadrightarrow G^{\text{ét}}$.
(3) f naturally induces homomorphisms $f^\circ : G^\circ \to H^\circ$ and $f^{\text{ét}} : G^{\text{ét}} \to H^{\text{ét}}$.

Proof. The first statement is evident since the image of G is a connected R-subgroup of H. The second statement follows from the fact that the image of G° lies in H° by the first statement and thus is trivial by Proposition 1.4.12. The last statement is an immediate consequence of the previous two statements. □

Proposition 1.4.14. Let G, G', and G'' be finite flat R-groups with a short exact sequence

$$\underline{0} \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow \underline{0}.$$

(1) The given exact sequence induces short exact sequences

$$\underline{0} \longrightarrow (G')^\circ \longrightarrow G^\circ \longrightarrow (G'')^\circ \longrightarrow \underline{0},$$
$$\underline{0} \longrightarrow (G')^{\text{ét}} \longrightarrow G^{\text{ét}} \longrightarrow (G'')^{\text{ét}} \longrightarrow \underline{0}.$$

(2) G is connected if and only if both G' and G'' are connected.
(3) G is étale if and only if both G' and G'' are étale.

Proof. Theorem 1.4.9 and Proposition 1.4.13 together yield a commutative diagram

$$\begin{array}{ccccccccc}
 & & \underline{0} & & \underline{0} & & \underline{0} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\underline{0} & \longrightarrow & (G')^\circ & \longrightarrow & G' & \longrightarrow & (G')^{\text{ét}} & \longrightarrow & \underline{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\underline{0} & \longrightarrow & G^\circ & \longrightarrow & G & \longrightarrow & G^{\text{ét}} & \longrightarrow & \underline{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\underline{0} & \longrightarrow & (G'')^\circ & \longrightarrow & G'' & \longrightarrow & (G'')^{\text{ét}} & \longrightarrow & \underline{0} \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \underline{0} & & \underline{0} & & \underline{0} & &
\end{array}$$

where the rows are exact. Since the middle column is exact, Proposition 1.4.12 implies that the right column is exact on the level of $\overline{k}$-points. We deduce from Proposition 1.3.4 that the right column is exact and consequently find by the snake lemma (or the nine lemma) that the left column is exact as well, thereby establishing statement (1). Statement (2) is an immediate consequence of Proposition 1.4.7. Statement (3) follows from statement (1) by Proposition 1.4.12. □

PROPOSITION 1.4.15. Assume that $R = k$ is a perfect field. For every finite flat k-group G, the connected-étale sequence canonically splits.

PROOF. Let G^{red} denote the reduction of G. If we write $G = \mathrm{Spec}\,(A)$ for some finite dimensional k-algebra A, we have $G^{\mathrm{red}} = \mathrm{Spec}\,(A^{\mathrm{red}})$ for $A^{\mathrm{red}} := A/\mathfrak{n}$ where $\mathfrak{n}$ denotes the nilradical of A. We wish to prove that the homomorphism $G \twoheadrightarrow G^{\text{ét}}$ admits a canonical section induced by the closed embedding $G^{\mathrm{red}} \hookrightarrow G$.

We assert that G^{red} is a k-subgroup of G. The scheme $G^{\mathrm{red}} \times_k G^{\mathrm{red}}$ is reduced by a general fact that the product of two reduced schemes over a perfect field is reduced as noted in the Stacks project [**Sta**, Tag 035Z]. Hence the image of $G^{\mathrm{red}} \times_k G^{\mathrm{red}}$ under the multiplication map lies in G^{red} by a standard fact stated in the Stacks project [**Sta**, Tag 0356]. Similarly, the image of G^{red} under the inverse map lies in G^{red}. In addition, the unit section of G factors through G^{red} as k is reduced. Therefore G^{red} is a k-subgroup of G as desired.

Let us now prove that G^{red} is finite étale. By construction, the affine ring A^{red} of G^{red} is a finite dimensional k-algebra. Hence we deduce from some general facts stated in the Stacks project [**Sta**, Tag 00J6 and Tag 00JB] that there exists a k-algebra isomorphism

$$A^{\mathrm{red}} \simeq \prod_{i=1}^{n} A_i^{\mathrm{red}}$$

where each A_i^{red} is a finite dimensional local k-algebra with a unique prime ideal. In fact, since A^{red} is reduced, each A_i^{red} is a finite field extension of k, which is separable as k is perfect. Now Lemma 1.3.2 implies that G^{red} is finite étale as desired.

It remains to show that the homomorphism $G^{\mathrm{red}} \hookrightarrow G \twoheadrightarrow G^{\text{ét}}$ is an isomorphism. The embedding $G^{\mathrm{red}} \hookrightarrow G$ induces an isomorphism $G^{\mathrm{red}}(\overline{k}) \cong G(\overline{k})$ as $\overline{k}$ is reduced. Moreover, the surjection $G \twoheadrightarrow G^{\text{ét}}$ induces an isomorphism $G(\overline{k}) \cong G^{\text{ét}}(\overline{k})$ as noted in Proposition 1.4.12. Therefore the homomorphism $G^{\mathrm{red}} \hookrightarrow G \twoheadrightarrow G^{\text{ét}}$ yields an isomorphism $G^{\mathrm{red}}(\overline{k}) \cong G^{\text{ét}}(\overline{k})$ which is clearly Γ_k-equivariant. Since G^{red} and $G^{\text{ét}}$ are both finite étale, we establish the desired assertion by Proposition 1.3.4. □

Example 1.4.16. We say that an elliptic curve E over $\overline{\mathbb{F}}_p$ is *ordinary* if $E[p](\overline{\mathbb{F}}_p)$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. We assert that every ordinary elliptic curve E over $\overline{\mathbb{F}}_p$ yields an isomorphism

$$E[p] \simeq \mu_p \times \underline{\mathbb{Z}/p\mathbb{Z}}.$$

Let us consider the connected-étale sequence

$$\underline{0} \longrightarrow E[p]^{\circ} \longrightarrow E[p] \longrightarrow E[p]^{\text{ét}} \longrightarrow \underline{0}. \tag{1.9}$$

We have $E[p]^{\text{ét}}(\overline{\mathbb{F}}_p) \simeq E[p](\overline{\mathbb{F}}_p) \simeq \mathbb{Z}/p\mathbb{Z}$ by Proposition 1.4.12 and thus find $E[p]^{\text{ét}} \simeq \underline{\mathbb{Z}/p\mathbb{Z}}$ by Proposition 1.3.7. Therefore the exact sequence (1.9) induces a dual exact sequence

$$\underline{0} \longrightarrow (\underline{\mathbb{Z}/p\mathbb{Z}})^{\vee} \longrightarrow E[p]^{\vee} \longrightarrow (E[p]^{\circ})^{\vee} \longrightarrow \underline{0}$$

by Proposition 1.2.13, where the second arrow is a closed embedding by Proposition 1.1.11. Now we apply Proposition 1.2.8 and Example 1.2.11 to identify the map $(\underline{\mathbb{Z}/p\mathbb{Z}})^{\vee} \hookrightarrow E[p]^{\vee}$ with a closed embedding $\mu_p \hookrightarrow E[p]$, which in turn gives rise to a closed embedding $\mu_p \hookrightarrow E[p]^{\circ}$ by Proposition 1.4.13 and Example 1.4.8. Moreover, as Example 1.1.15 and Proposition 1.1.16 show that $E[p]^{\text{ét}} \simeq \underline{\mathbb{Z}/p\mathbb{Z}}$ and $E[p]$ respectively have order p and p^2, Theorem 1.1.18 implies that $E[p]^{\circ}$ has order $p^2/p = p$. Since μ_p also has order p by Example 1.1.15, the closed embedding $\mu_p \hookrightarrow E[p]^{\circ}$ is indeed an isomorphism by Theorem 1.1.18. Hence we obtain the desired isomorphism by Proposition 1.4.15.

1.5. The Frobenius morphism

For this subsection, we assume that $R = k$ is a field of characteristic p and write σ for the Frobenius endomorphism of k. We introduce and study homomorphisms of finite flat k-groups induced by σ.

Definition 1.5.1. Let $T = \mathrm{Spec}\,(B)$ be an affine k-scheme and r be a positive integer.

(1) The p^r*-Frobenius twists* of B and T are respectively

$$B^{(p^r)} := B \otimes_{k,\sigma^r} k \quad \text{ and } \quad T^{(p^r)} := T \times_{k,\sigma^r} k = \mathrm{Spec}\,(B^{(p^r)}),$$

where the factor k in the products has σ^r as structure morphism.

(2) The *relative* p^r*-Frobenius* of B is the k-algebra homomorphism $\varphi_B^{[r]} : B^{(p^r)} \to B$ which maps each $b \otimes c \in B^{(p^r)} = B \otimes_{k,\sigma^r} k$ to $c \cdot b^{p^r} \in B$.

(3) The *relative* p^r*-Frobenius* of T is the morphism $\varphi_T^{[r]} : T \to T^{(p^r)}$ induced by $\varphi_B^{[r]}$.

(4) For $r = 1$, we often refer to $\varphi_B := \varphi_B^{[1]}$ and $\varphi_T := \varphi_T^{[1]}$ as the *Frobenii* of B and T.

Remark. We can similarly define the Frobenius twists and relative Frobenii for all k-schemes.

LEMMA 1.5.2. Let $T = \mathrm{Spec}\,(B)$ be an affine k-scheme and r be a positive integer.

(1) The Frobenius twists satisfy recursive relations

$$B^{(p^{r+1})} = \left(B^{(p^r)}\right)^{(p)} \quad \text{ and } \quad T^{(p^{r+1})} = \left(T^{(p^r)}\right)^{(p)}.$$

(2) The relative Frobenii satisfy recursive relations

$$\varphi_B^{[r+1]} = \varphi_B^{[r]} \circ \varphi_{B^{(p^r)}} \quad \text{ and } \quad \varphi_T^{[r+1]} = \varphi_{T^{(p^r)}} \circ \varphi_T^{[r]}.$$

PROOF. The assertions are evident by definition. □

PROPOSITION 1.5.3. Let $T = \mathrm{Spec}\,(B)$ be a k-variety with $B = k[t_1, \cdots, t_n]/(f_1, \cdots, f_m)$ for some polynomials $f_1, \cdots, f_m$ in n variables. Fix a positive integer r.

(1) There exists a canonical k-algebra isomorphism

$$B^{(p^r)} \cong k[t_1, \cdots, t_n]/(f_1^{(p^r)}, \cdots, f_m^{(p^r)})$$

with $f_i^{(p^r)}$ obtained from f_i by raising each coefficient to the p^r-th power.

(2) The homomorphism $\varphi_B^{[r]}$ maps each $t_i \in B^{(p^r)}$ to $t_i^{p^r} \in B$.

(3) For a $\overline{k}$-point on T that represents a common root $(c_1, \cdots, c_n)$ of $f_1, \cdots, f_m$, its image under $\varphi_T^{[r]}$ represents the common root $(c_1^{p^r}, \cdots, c_n^{p^r})$ of $f_1^{(p^r)}, \cdots, f_m^{(p^r)}$.

PROOF. Statement (1) follows from the fact that under the canonical identification $k[t_1, \cdots, t_n]^{(p^r)} \cong k[t_1, \cdots, t_n]$, the natural map $k[t_1, \cdots, t_n] \to k[t_1, \cdots, t_n]^{(p^r)}$ raises the coefficients of each polynomial to their p^r-th powers. Statement (2) follows immediately from statement (1). Statement (3) is a straightforward consequence of statement (2). □

PROPOSITION 1.5.4. Given an affine k-scheme $T = \mathrm{Spec}\,(B)$ and a positive integer r, the morphism $\varphi_T^{[r]}$ induces a natural bijection $T(\overline{k}) \cong T^{(p^r)}(\overline{k})$.

PROOF. Let $\mathrm{Frob}_T : T \to T$ denote the morphism induced by the p-th power map on B. Under the natural bijection $T^{(p^r)}(\overline{k}) = T(\overline{k}) \times (\mathrm{Spec}\,(k))\,(\overline{k}) \cong T(\overline{k})$ given by the fact that $(\mathrm{Spec}\,(k))\,(\overline{k})$ is a singleton, $\varphi_T^{[r]}$ maps each $t \in T(\overline{k})$ to $\mathrm{Frob}_T^r(t)$ by construction. Hence we establish the desired assertion by observing that Frob_T^r induces a bijection $T(\overline{k}) \cong T(\overline{k})$. □

Definition 1.5.5. Given a morphism $f : T \to U$ of affine k-schemes and a positive integer r, we refer to the induced morphism $f^{(p^r)} : T^{(p^r)} \to U^{(p^r)}$ as the *p^r-Frobenius twist* of f.

Example 1.5.6. Given an arbitrary affine k-scheme $T = \mathrm{Spec}\,(B)$, we show the equality

$$\big(\varphi_T^{[r]}\big)^{(p^s)} = \varphi_{T^{(p^s)}}^{[r]}$$

for any positive integers r and s. For $r = 1$ and $s = 1$, since we have a commutative diagram

$$\begin{array}{ccccc} T^{(p)} & \xrightarrow{(\varphi_T)^{(p)}} & T^{(p^2)} & \longrightarrow & \mathrm{Spec}\,(k) \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\ T & \xrightarrow{\varphi_T} & T^{(p)} & \longrightarrow & \mathrm{Spec}\,(k) \end{array}$$

where each square is cartesian, we find $(\varphi_T)^{(p)} = \varphi_{T^{(p)}}$ by observing that the morphism $T^{(p)} \longrightarrow T \xrightarrow{\varphi_T} T^{(p)}$ given by the left square induces the p-th power map on $B^{(p)}$. For $r = 1$ and $s \geq 2$, we have $(\varphi_T)^{(p^s)} = \big((\varphi_T)^{(p^{s-1})}\big)^{(p)}$ and thus proceed by induction to find $(\varphi_T)^{(p^s)} = \varphi_{T^{(p^s)}}$. Finally, for $r \geq 2$ and $s \geq 2$, we have

$$\big(\varphi_T^{[r]}\big)^{(p^s)} = \big(\varphi_{T^{(p^{r-1})}} \circ \varphi_T^{[r-1]}\big)^{(p^s)} = \big(\varphi_{T^{(p^{r-1})}}\big)^{(p^s)} \circ \big(\varphi_T^{[r-1]}\big)^{(p^s)}$$

by Lemma 1.5.2 and thus proceed by induction to obtain the desired equality.

Lemma 1.5.7. Let T and U be affine k-schemes. Take a positive integer r.

(1) There exists a natural isomorphism $(T \times_k U)^{(p^r)} \cong T^{(p^r)} \times_k U^{(p^r)}$ which canonically identifies $\varphi_{(T \times_k U)}^{[r]}$ with $\varphi_T^{[r]} \times_k \varphi_U^{[r]}$.

(2) Every k-scheme morphism $f : T \to U$ gives rises to a commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\varphi_T^{[r]}} & T^{(p^r)} \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f^{(p^r)}} \\ U & \xrightarrow{\varphi_U^{[r]}} & U^{(p^r)} \end{array}$$

where all maps are k-scheme morphisms.

Proof. The assertions are straightforward to verify using properties of fiber products. □

Proposition 1.5.8. Let G be an affine k-group and r be a positive integer.

(1) The p^r-Frobenius twist $G^{(p^r)}$ is naturally an affine k-group.

(2) The relative p^r-Frobenius $\varphi_G^{[r]}$ is a k-group homomorphism.

(3) If G is finite flat, $G^{(p^r)}$ is finite flat with a natural isomorphism $\big(G^{(p^r)}\big)^\vee \cong (G^\vee)^{(p^r)}$.

Proof. As we have $G^{(p^r)} = G \times_{k,\sigma^r} k$, statements (1) and (3) are evident by Lemma 1.2.1 and Proposition 1.2.5. Statements (2) is a straightforward consequence of Lemma 1.5.7. □

Lemma 1.5.9. Let $f : G \to H$ be a homomorphism of affine k-groups.

(1) The p^r-Frobenius twist $f^{(p^r)}$ is a k-group homomorphism for each $r \geq 1$.

(2) If f is a closed embedding, $f^{(p^r)}$ is also a closed embedding for each $r \geq 1$.

(3) If f is an isomorphism, $f^{(p^r)}$ is also an isomorphism for each $r \geq 1$.

Proof. The first statement is striaghtforward to verify by Lemma 1.5.7. The remaining statements are evident by the construction of the Frobenius twists via base changes. □

Definition 1.5.10. Let G be a finite flat k-group and r be a positive integer.

(1) We define the p^r*-Verschiebung* to be $\psi_G^{[r]} := \left(\varphi_{G^\vee}^{[r]}\right)^\vee$, regarded as a homomorphism from $G^{(p^r)} \cong \left((G^\vee)^{(p^r)}\right)^\vee$ to $G \cong (G^\vee)^\vee$.

(2) For $r = 1$, we often refer to $\psi_G := \psi_G^{[1]} = \varphi_{G^\vee}^\vee$ as the *Verschiebung* of G.

PROPOSITION 1.5.11. We identify the Frobenius and Verschiebung of $\alpha_p,\ \mu_p,\ \underline{\mathbb{Z}/p\mathbb{Z}}$ as follows:

(1) For α_p, we have $\varphi_{\alpha_p} = 0$ and $\psi_{\alpha_p} = 0$.
(2) For μ_p, we have $\varphi_{\mu_p} = 0$ and $\psi_{\mu_p} = \mathrm{id}_{\mu_p}$.
(3) For $\underline{\mathbb{Z}/p\mathbb{Z}}$, we have $\varphi_{\underline{\mathbb{Z}/p\mathbb{Z}}} = \mathrm{id}_{\underline{\mathbb{Z}/p\mathbb{Z}}}$ and $\psi_{\underline{\mathbb{Z}/p\mathbb{Z}}} = 0$.

PROOF. Let us begin with the Frobenii. We use the affine descriptions in Example 1.1.7. For α_p, we find $\alpha_p^{(p)} \cong \alpha_p$ and $\varphi_{\alpha_p} = 0$ by Proposition 1.5.3. For μ_p, we similarly find $\mu_p^{(p)} \cong \mu_p$ and $\varphi_{\mu_p} = 0$. Let us now consider $\underline{\mathbb{Z}/p\mathbb{Z}}$. We write $A := \prod\limits_{i \in \mathbb{Z}/p\mathbb{Z}} k$ for its affine ring and e_i for the element of A whose only nonzero entry is 1 in the component corresponding to i. Since the A-algebra $A^{(p)}$ admits a natural identification

$$A^{(p)} = \left(\prod_{i \in \mathbb{Z}/p\mathbb{Z}} k\right) \otimes_{k,\sigma} k \cong \prod_{i \in \mathbb{Z}/p\mathbb{Z}} (k \otimes_{k,\sigma} k) \cong \prod_{i \in \mathbb{Z}/p\mathbb{Z}} k = A,$$

for each $a = \sum\limits_{i \in \mathbb{Z}/p\mathbb{Z}} c_i e_i \in A$ with $c_i \in k$ we find

$$\varphi_A(a) = \varphi_A\left(\sum_{i \in \mathbb{Z}/p\mathbb{Z}} c_i e_i\right) = \sum_{i \in \mathbb{Z}/p\mathbb{Z}} \varphi_A(c_i e_i) = \sum_{i \in \mathbb{Z}/p\mathbb{Z}} c_i e_i^p = \sum_{i \in \mathbb{Z}/p\mathbb{Z}} c_i e_i = a.$$

Hence $\varphi_{\underline{\mathbb{Z}/p\mathbb{Z}}}$ coincides with the identity map. Now that we have the desired identifications of the Frobenii, we deduce the identifications for the Verschiebungs from the results on Cartier duals such as Proposition 1.2.8 and Proposition 1.2.9. $\square$

LEMMA 1.5.12. Given a finite flat k-group G, we have $\psi_G^{[r+1]} = \psi_G^{[r]} \circ \psi_{G^{(p^r)}}$ for each $r \geq 1$.

PROOF. The assertion is evident by Lemma 1.5.2. $\square$

LEMMA 1.5.13. Let G and H be finite flat k-group schemes. Take a positive integer r.

(1) There exists a natural isomorphism $(G \times_k H)^{(p^r)} \cong G^{(p^r)} \times_k H^{(p^r)}$ which canonically identifies $\psi_{(G \times_k H)}^{[r]}$ with $\psi_G^{[r]} \times_k \varphi_H^{[r]}$.
(2) Every homorphism $f : G \to H$ of finite flat k-groups induces commutative diagrams

$$\begin{array}{ccc} G & \xrightarrow{\varphi_G^{[r]}} & G^{(p^r)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{(p^r)}} \\ H & \xrightarrow{\varphi_H^{[r]}} & H^{(p^r)} \end{array} \qquad \begin{array}{ccc} G & \xleftarrow{\psi_G^{[r]}} & G^{(p^r)} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f^{(p^r)}} \\ H & \xleftarrow{\psi_H^{[r]}} & H^{(p^r)} \end{array}$$

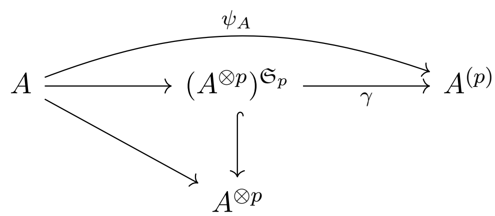
where all maps are k-group homomorphisms.

PROOF. By Lemma 1.2.1, fiber products of finite flat k-groups are finite flat k-groups. Hence the assertions follow from Lemma 1.5.7, Proposition 1.5.8, and Lemma 1.5.9. $\square$

PROPOSITION 1.5.14. Let $G = \operatorname{Spec}(A)$ be a finite flat k-group. We denote the symmetric group of order p by $\mathfrak{S}_p$, which acts on $A^{\otimes p}$ by permuting factors of pure tensors.

(1) There exists a k-algebra homomorphism $\gamma : (A^{\otimes p})^{\mathfrak{S}_p} \to A^{(p)}$ with the following properties:
   (i) For each $a \in A$, we have $\gamma(a^{\otimes p}) = a \otimes 1$.
   (ii) For each pure tensor in $A^{\otimes p}$ with unequal factors, the sum of elements in its $\mathfrak{S}_p$-orbit maps to 0 under γ.

(2) The k-algebra homomorphism ψ_A induced by ψ_G fits into a commutative diagram



with the map $A \to A^{\otimes p}$ induced by the comultiplication of G.

PROOF. Let us work with the natural k-algebra isomorphisms

$$A \cong (A^\vee)^\vee, \quad \left(\operatorname{Sym}^p A^\vee\right)^\vee \cong (A^{\otimes p})^{\mathfrak{S}_p}, \quad A^{(p)} \cong \left((A^\vee)^{(p)}\right)^\vee,$$

given by Theorem 1.2.4, Proposition 1.5.8, and the fact that $\operatorname{Sym}^p(A^\vee)$ is the k-algebra of $\mathfrak{S}_p$-covariants for $(A^\vee)^{\otimes p}$. Since k has characteristic p, we have $(f_1 + f_2)^{\otimes p} = f_1^{\otimes p} + f_2^{\otimes p}$ in $\operatorname{Sym}^p(A^\vee)$ for any $f_1,\ f_2 \in A^\vee$. Therefore there exists a unique k-algebra homomorphism $\theta : (A^\vee)^{(p)} \to \operatorname{Sym}^p A^\vee$ which maps each $f \otimes c \in (A^\vee)^{(p)} = A^\vee \otimes_{k,\sigma} k$ to $c \cdot f^{\otimes p} \in \operatorname{Sym}^p A^\vee$. Let us take γ to be the dual of θ. In addition, we identify each $a \in A$ with its image $\mathfrak{e}_a$ under the isomorphism $A \cong (A^\vee)^\vee$. For each $a \in A$ and $f \otimes c \in (A^\vee)^{(p)} = A^\vee \otimes_{k,\sigma} k$, we have

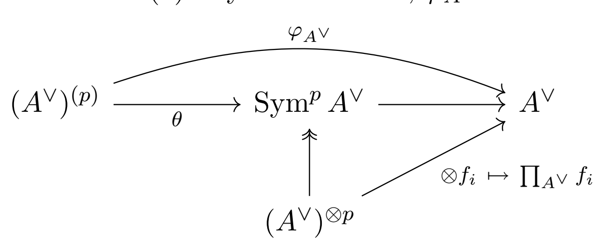
$$\gamma(a^{\otimes p})(f \otimes c) = (\mathfrak{e}_a)^{\otimes p}(c \cdot f^{\otimes p}) = c \cdot f(a)^p = (\mathfrak{e}_a \otimes 1)(f \otimes c)$$

where the last equality follows from the identity $f(a) \otimes c = 1 \otimes (c \cdot f(a)^p)$ in $A \otimes_{k,\sigma} k$. Moreover, given a pure tensor $\otimes a_i \in A^{\otimes p}$ with unequal factors, we denote its $\mathfrak{S}_p$-stabilizer by S and find

$$\gamma\left(\sum_{\tau \in \mathfrak{S}_p/S} \bigotimes_{i=1}^{p} a_{\tau(i)}\right)(f \otimes c) = \sum_{\tau \in \mathfrak{S}_p/S} \left(\bigotimes_{i=1}^{p} \mathfrak{e}_{a_{\tau(i)}}\right)(c \cdot f^{\otimes p}) = c \sum_{\tau \in \mathfrak{S}_p/S} \prod_{i=1}^{p} f(a_i) = 0$$

for each $f \otimes c \in (A^\vee)^{(p)} = A^\vee \otimes_{k,\sigma} k$, where the last equality follows from the fact that the number of elements in $\mathfrak{S}_p/S$ is divisible by p. Therefore we establish statement (1).

Let us now consider statement (2). By construction, $\varphi_{A^\vee}$ fits into a commutative diagram

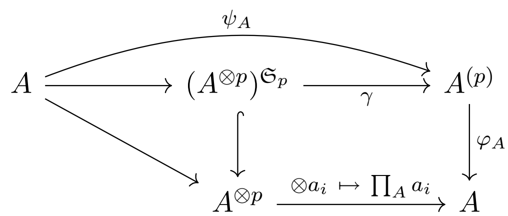


where $\prod_{A^\vee}$ denotes the ring multiplication on $A^\vee$. Theorem 1.2.4 implies that the dual of the map $(A^\vee)^{\otimes p} \to A^\vee$ in the diagram coincides with the map $A \to A^{\otimes p}$ induced by the comultiplication of G. Since we have $\psi_A = \varphi_{A^\vee}^\vee$ by construction, we obtain the diagram in statement (2) by dualizing the above diagram, thereby completing the proof. $\square$

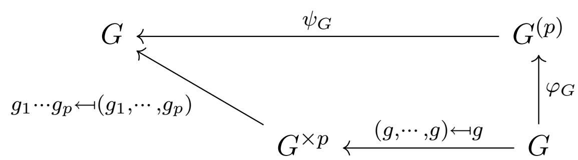
Proposition 1.5.15. Every finite flat k-group G yields the identities

$$\psi_G^{[r]} \circ \varphi_G^{[r]} = [p^r]_G \quad \text{and} \quad \varphi_G^{[r]} \circ \psi_G^{[r]} = [p^r]_{G^{(p)}} \quad \text{for each integer } r \geq 1.$$

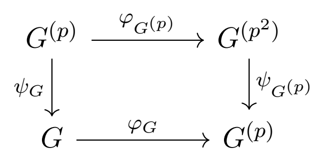
Proof. An inductive argument based on Lemma 1.5.2 and Lemma 1.5.12 shows that it suffices to establish the desired identities for $r = 1$. Let us write $G = \operatorname{Spec}(A)$ for some finite dimensional k-algebra A. In addition, we let ψ_A denote the k-algebra homomorphism induced by ψ_G and $\mathfrak{S}_p$ denote the symmetric group of order p. Proposition 1.5.14 yields a commutative diagram



with the diagonal map $A \to A^{\otimes p}$ and the bottom horizontal map $A^{\otimes p} \to A$ respectively induced by the comultiplication of G and the multiplication of A. Therefore we have a commutative diagram



and in turn find $\psi_G \circ \varphi_G = [p]_G$. Moreover, we have $\varphi_G^{(p)} = \varphi_{G^{(p)}}$ as noted in Example 1.5.6 and thus obtain a commutative diagram



by Lemma 1.5.13. Since we have established the identity $\psi_G \circ \varphi_G = [p]_G$ for an arbitrary finite flat k-group G, we find $\varphi_G \circ \psi_G = \psi_{G^{(p)}} \circ \varphi_{G^{(p)}} = [p]_{G^{(p)}}$ as desired, thereby completing the proof. $\square$

Remark. Let us briefly discuss the Verschiebung for a general affine k-group $G = \operatorname{Spec}(A)$ which is not necessarily finite flat. Our proof of Proposition 1.5.14 readily shows that statement (1) holds for an arbitrary k-algebra A. In addition, the associativity axiom for G implies that the k-algebra homomorphism $A \to A^{\otimes p}$ induced by the comultiplication of G factors through the embedding $(A^{\otimes p})^{\mathfrak{S}_p} \hookrightarrow A^{\otimes p}$. Therefore there exists a unique k-algebra homomorphism $\psi_A : A \to A^{(p)}$ which fits into the diagram in statement (2). We define the Verschiebung of G to be the k-scheme morphism $\psi_G : G^{(p)} \to G$ induced by ψ_A. It is not hard to verify that ψ_A is compatible with the comultiplications, which means that ψ_G is a k-group homomorphism. Moreover, for each integer $r \geq 1$ we inductively define the k-group homomorphism $\psi_G^{[r]}$ by the recursive relation in Lemma 1.5.12. It turns out that Lemma 1.5.13 and Proposition 1.5.15 hold for general affine k-groups; indeed, we can establish Lemma 1.5.13 by a straightforward argument on affine rings and in turn deduce Proposition 1.5.15 by the same proof. In addition, we can suitably adjust our argument in Example 1.5.6 to obtain the identity $(\psi_G^{[r]})^{(p^s)} = \psi_{G^{(p^s)}}^{[r]}$ for any positive integers r and s.

LEMMA 1.5.16. Let $G = \mathrm{Spec}\,(A)$ be a finite flat k-group.

(1) The Frobenius φ_G is an isomorphism if and only if it is injective.

(2) If G is connected, A is an artinian local k-algebra with its maximal ideal given by the augmentation ideal of G.

PROOF. Since G and $G^{(p)}$ are of the same order by construction, statement (1) follows from Proposition 1.1.11 and Theorem 1.1.18. If G is connected, A is an artinian local ring by Lemma 1.1.14, Lemma 1.4.2, and a general fact that every finite dimensional algebra over a field is artinian as noted in the Stacks project [**Sta**, Tag 00J6]. Hence we deduce statement (2) by observing that the augmentation ideal I of G is a maximal ideal as we have $A/I \cong k$. $\square$

PROPOSITION 1.5.17. Let $G = \mathrm{Spec}\,(A)$ be a finite flat k-group.

(1) G is connected if and only if $\varphi_G^{[r]}$ vanishes for some integer $r \geq 1$.

(2) G is étale if and only if φ_G is an isomorphism.

PROOF. Let us begin with statement (1). If $\varphi_G^{[r]}$ vanishes for some $r \geq 1$, we find by Proposition 1.5.4 that $G(\overline{k})$ is trivial and thus deduce from Proposition 1.4.7 that G is connected. For the converse, we now assume that G is connected. Its augmentation ideal I is nilpotent by Lemma 1.5.16 and a standard fact stated in the Stacks project [**Sta**, Tag 00J8]; in particular, there exists an integer $r \geq 1$ with $t^{p^r} = 0$ for all $t \in I$. Therefore $\varphi_A^{[r]}$ factors through the surjection $A^{(p^r)} = A \otimes_{k,\sigma^r} k \twoheadrightarrow (A/I) \otimes_{k,\sigma^r} k$ induced by the unit section of $G^{(p^r)}$. We deduce that $\varphi_G^{[r]}$ vanishes and in turn establish statement (1).

It remains to prove statement (2). Let us assume that φ_G is an isomorphism. It is not hard to see that φ_{G° is an isomorphism, for example by Lemma 1.5.7 and Lemma 1.5.16. Hence Example 1.5.6 and Lemma 1.5.9 together imply that $\varphi_{(G^\circ)^{(p^r)}} = \varphi_{G^\circ}^{(p^r)}$ is an isomorphism for each $r \geq 1$. Now a simple induction based on Lemma 1.5.2 shows that $\varphi_{G^\circ}^{[r]}$ is an isomorphism for each $r \geq 1$. Since $\varphi_{G^\circ}^{[r]}$ vanishes for some $r \geq 1$ by statement (1), we find that G° is trivial and consequently deduce from Proposition 1.4.12 that G is étale.

We now assume for the converse that G is étale. It is not hard to see by Lemma 1.5.7 that $\varphi_{\ker(\varphi_G)}$ vanishes. Hence statement (1) implies that $\ker(\varphi_G)$ is connected, which means that $\ker(\varphi_G)$ lies in G°. Since G° is trivial by Proposition 1.4.12, we find that $\ker(\varphi_G)$ is trivial and in turn deduce from Lemma 1.5.16 that φ_G is an isomorphism. $\square$

Remark. Proposition 1.5.17 yields similar criteria for $G^\vee$ to be connected or étale in terms of the Verschiebungs.

Example 1.5.18. Let E be an ordinary elliptic curve over $\overline{\mathbb{F}}_p$. We assert that there exists an isomorphism $\ker(\varphi_{E[p]}) \simeq \mu_p$. Example 1.4.16 shows that we have $E[p]^\circ \simeq \mu_p$. Lemma 1.5.7 and Proposition 1.5.17 together imply that $\ker(\varphi_{E[p]})$ is connected and thus lies in $E[p]^\circ \simeq \mu_p$. On the other hand, $\ker(\varphi_{E[p]})$ contains $E[p]^\circ \simeq \mu_p$ as φ_{μ_p} vanishes by Proposition 1.5.11. Therefore we have $\ker(\varphi_{E[p]}) = E[p]^\circ \simeq \mu_p$ as desired.

Remark. As noted after Definition 1.5.1, we can define the relative Frobenii for general k-schemes, including abelian k-varieties. Moreover, since abelian varieties admit a notion of duality, we can define their relative Verschiebungs as in Definition 1.5.10. It turns out that most results that in this subsection remain valid for abelian varieties. In particular, for an ordinary elliptic curve E over $\overline{\mathbb{F}}_p$, we find $\ker(\varphi_E) \subseteq E[p]$ by the identity $\psi_E^{[r]} \circ \varphi_E^{[r]} = [p^r]_E$ and in turn obtain an isomorphsim $\ker(\varphi_E) \simeq \mu_p$ from Example 1.5.18.

PROPOSITION 1.5.19. Let $G = \operatorname{Spec}(A)$ be a finite flat k-group with augmentation ideal I.

(1) For each integer $r \geq 1$, there exists a natural isomorphism
$$\ker(\varphi_G^{[r]}) \cong \operatorname{Spec}(A/I^{(p^r)})$$
where $I^{(p^r)}$ denotes the ideal generated by the p^r-th powers of elements in I.

(2) If φ_G vanishes, the order of G is p^d where d denotes the dimension of I/I^2 over k.

PROOF. Let us denote by e the unit section of G, which we naturally identify with the closed embedding $\operatorname{Spec}(A/I) \hookrightarrow \operatorname{Spec}(A)$. The unit section of $G^{(p^r)}$ is $e^{(p^r)}$, induced by natural surjection $A^{(p^r)} = A \otimes_{k,\sigma^r} k \twoheadrightarrow (A/I) \otimes_{k,\sigma^r} k$. Hence statement (1) follows from the identification of $\ker(\varphi_G^{[r]})$ as the fiber of $\varphi_G^{[r]}$ over $e^{(p^r)}$.

It remains to establish statement (2). We chooose $a_1, \cdots, a_d \in I$ whose images in I/I^2 form a basis over k. Since G is connected by Proposition 1.5.17, we note by Lemma 1.5.16 that A is a local ring with maximal ideal I and in turn deduce from Nakayama's lemma that $a_1, \cdots, a_d$ generate I. Therefore statement (1) yields an isomorphism $A \cong A/(a_1^p, \cdots, a_d^p)$. Let us take the k-algebra homomorphism
$$\lambda : k[t_1, \cdots, t_d] \longrightarrow A \cong A/(a_1^p, \cdots, a_d^p)$$
which maps each t_i to a_i. It is not hard to see by Lemma 1.1.10 that λ is surjective, which means that λ yields an isomorphism
$$k[t_1, \cdots, t_d]/\ker(\lambda) \simeq A.$$
Hence $\Omega_{A/k}$ admits an isomorphism
$$\Omega_{A/k} \simeq \bigoplus_{i=1}^{d} A \cdot dt_i \Big/ \sum_{f \in \ker(\lambda)} A \cdot df$$
by a standard fact about differentials stated in the Stacks project [**Sta**, Tag 00RU]. Since $\Omega_{A/k}$ is a free A-module of rank d by Proposition 1.3.5, we see that $\sum\limits_{f \in \ker(\lambda)} A \cdot df$ is zero and in turn find that $\ker(\lambda)$ is stable under partial derivatives. If $\ker(\lambda)$ is not a subset of $(t_1^p, \cdots, t_d^p)$, we take a nonzero element $f \in \ker(\lambda) \backslash (t_1^p, \cdots, t_d^p)$ with minimal degree and observe that its partial derivatives yield elements in $\ker(\lambda)$, which contradicts the minimality for f. Now we must have $\ker(\lambda) = (t_1^p, \cdots, t_d^p)$, as $\ker(\lambda)$ evidently contains $(t_1^p, \cdots, t_d^p)$, and consequently deduce that $A \simeq k[t_1, \cdots, t_d]/(t_1^p, \cdots, t_d^p)$ is free of dimension p^d over k, thereby establishing statement (2). □

PROPOSITION 1.5.20. If a finite flat k-group G is connected, its order is a power of p.

PROOF. Let us denote the order of G by n. Since the assertion is trivial for $n = 1$, we henceforth assume the inequality $n > 1$ and proceed by induction on n. It is evident by Proposition 1.4.12 that G is not étale. Hence Lemma 1.5.16 and Proposition 1.5.17 together imply that $\ker(\varphi_G)$ is not trivial. In addition, as Proposition 1.1.11 implies that $\ker(\varphi_G)$ is a closed k-subgroup of G, we apply Proposition 1.4.14 to see that both $\ker(\varphi_G)$ and $G/\ker(\varphi_G)$ are connected. Let us write n_1 and n_2 respectively for the orders of $\ker(\varphi_G)$ and $G/\ker(\varphi_G)$. By Theorem 1.1.18 we have $n = n_1 n_2$. If φ_G does not vanish, we find that both n_1 and n_2 are less than n and thus are powers of p by the induction hypothesis, which in particular implies that n is a power of p. If φ_G vanishes, Proposition 1.5.19 shows that n is a power of p. Hence we establish the desired assertion. □

Proposition 1.5.21. Given a finite flat k-group $G = \mathrm{Spec}\,(A)$ with unit section e, its tangent space at e admits a canonical isomorphism $t_{G,e} \cong \mathrm{Hom}_{k\text{-grp}}(G^\vee, \mathbb{G}_a)$.

Proof. Let us write I for the augmentation ideal of G and regard the unit section e as a k-point of G via the natural closed embedding $\mathrm{Spec}\,(k) \cong \mathrm{Spec}\,(A/I) \hookrightarrow \mathrm{Spec}\,(A)$. The tangent space $t_{G,e}$ is by definition canonically isomorphic to the kernel of the natural homomorphism $G(k[t]/(t^2)) \to G(k)$, which we naturally identify with the group of k-algebra homomorphisms $A \to k[t]/(t^2)$ whose composition with the map $k[t]/(t^2) \to k$ equals the counit ϵ of G. Since we can uniquely write every k-linear map $A \to k[t]/(t^2)$ in the form $f_0 + tf_1$ with $f_0,\ f_1 \in A^\vee = \mathrm{Hom}_{k\text{-mod}}(A, k)$, we find

$$\begin{aligned} t_{G,e} &\cong \left\{ f \in \mathrm{Hom}_{k\text{-alg}}(A, k[t]/(t^2)) : f = \epsilon + tg \text{ with } g \in A^\vee \right\} \\ &\cong \left\{ g \in A^\vee : \epsilon + tg \in \mathrm{Hom}_{k\text{-alg}}(A, k[t]/(t^2)) \right\}. \end{aligned}$$

For each $g \in A^\vee$, we have $\epsilon + tg \in \mathrm{Hom}_{k\text{-alg}}(A, k[t]/(t^2))$ if and only if it satisfies the identities

$$\epsilon(ab) + tg(ab) = (\epsilon(a) + tg(a))\,(\epsilon(b) + tg(b)) \quad \text{and} \quad \epsilon(1) + tg(1) = 1 \quad \text{for each } a,\ b \in A,$$

which are equivalent to the identities

$$g(ab) = \epsilon(a)g(b) + \epsilon(b)g(a) \quad \text{and} \quad g(1) = 0 \quad \text{for each } a,\ b \in A$$

by the fact that ϵ is an k-algebra homomorphism. We observe that the second identity is redundant as it follows from the first identity for $a = b = 1$. In addition, the first identity is equivalent to the commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad g \quad} & k \cong k \otimes_k k \\ {\scriptstyle m_A}\uparrow & \nearrow_{\epsilon \otimes g + g \otimes \epsilon} & \\ A \otimes_k A & & \end{array}$$

where m_A denotes the ring multiplication map on A. We dualize this diagram under the identification $A^\vee = \mathrm{Hom}_{k\text{-mod}}(A, k) \cong \mathrm{Hom}_{k\text{-mod}}(k, A^\vee)$ and find $m_A^\vee(g) = g \otimes 1 + 1 \otimes g$. Therefore we obtain a natural isomorphsim

$$t_{G,e} \cong \left\{ g \in A^\vee : m_A^\vee(g) = g \otimes 1 + 1 \otimes g \right\}.$$

Meanwhile, by Example 1.1.7 and Theorem 1.2.4 we find

$$\mathrm{Hom}_{k\text{-grp}}(G^\vee, \mathbb{G}_a) \cong \left\{ f \in \mathrm{Hom}_{k\text{-alg}}(k[t], A^\vee) : m_A^\vee(f(t)) = f(t) \otimes 1 + 1 \otimes f(t) \right\}$$

where the identity $m_A^\vee(f(t)) = f(t) \otimes 1 + 1 \otimes f(t)$ comes from compatibility with comultiplications. Since we have the canonical isomorphism $\mathrm{Hom}_{k\text{-alg}}(k[t], A^\vee) \cong A^\vee$ which sends each $f \in \mathrm{Hom}_{k\text{-alg}}(k[t], A^\vee)$ to $f(t)$, we obtain a natural identification

$$\mathrm{Hom}_{k\text{-grp}}(G^\vee, \mathbb{G}_a) \cong \left\{ g \in A^\vee : m_A^\vee(g) = g \otimes 1 + 1 \otimes g \right\}.$$

Therefore we deduce the desired assertion, thereby completing the proof. □

Proposition 1.5.22. A finite flat k-group G is étale if and only if $\mathrm{Hom}_{k\text{-grp}}(G^\vee, \mathbb{G}_a)$ vanishes.

Proof. Let us write $G = \mathrm{Spec}\,(A)$ for some finite dimensional k-algebra A. We denote the augmentation ideal of G by I and regard the unit section e as a k-point of G via the closed embedding $\mathrm{Spec}\,(k) \cong \mathrm{Spec}\,(A/I) \hookrightarrow \mathrm{Spec}\,(A)$. The tangent space $t_{G,e}$ is naturally isomorphic to the dual of I/I^2 by a general fact stated in the Stacks project [**Sta**, Tag 0B2E]. Therefore, by Proposition 1.3.5, G is étale if and only if $t_{G,e}$ vanishes. Now the desired assertion follows from Proposition 1.5.21. □

THEOREM 1.5.23. Assume that k is algebraically closed.

(1) Every simple finite flat k-group is either étale or connected.
(2) The simple finite étale k-groups are $\underline{\mathbb{Z}/\ell\mathbb{Z}}$ where ℓ ranges over all prime numbers.
(3) The simple connected finite flat k-groups are μ_p and α_p.

PROOF. Statement (1) is straightforward to verify by Theorem 1.4.9. Statement (2) follows from Proposition 1.3.6, Proposition 1.3.7, and the fact that the simple abelian groups are precisely the cyclic groups of prime order. Hence it remains to prove statement (3).

The k-groups μ_p and α_p are indeed connected as noted in Example 1.4.8. Moreover, they are of order p by construction and thus are simple by Theorem 1.4.9. We wish to show that they are the only simple connected finite flat k-groups.

Let G be a simple connected finite flat k-group. Theorem 1.2.4 and Proposition 1.2.13 together imply that $G^\vee$ is simple. Hence $G^\vee$ is either étale or connected by statement (1).

We consider the case where $G^\vee$ is étale. Statement (2) yields an isomorphism $G^\vee \simeq \underline{\mathbb{Z}/\ell\mathbb{Z}}$ for some prime ℓ. Hence G has order ℓ by Example 1.1.15 and Theorem 1.2.4. On the other hand, the order of G is a power of p as noted in Proposition 1.5.20. We thus find $\ell = p$ and in turn obtain an isomorphism $G \simeq \mu_p$ by Proposition 1.2.8.

Let us now consider the case where $G^\vee$ is connected. It is evident by Proposition 1.4.12 that neither G nor $G^\vee$ is étale. Theorem 1.2.4 and Proposition 1.5.22 together yield a nonzero k-group homomorphism $f : G \to \mathbb{G}_a$, which is indeed a closed embedding as G is simple. Moreover, Lemma 1.5.16 and Proposition 1.5.17 together imply that $\ker(\varphi_G)$ is not trivial, which means that φ_G vanishes as G is simple. Therefore f must factor through $\ker(\varphi_{\mathbb{G}_a})$, which is isomorphic to α_p as easily seen by Example 1.1.7 and Proposition 1.5.3. Since α_p is simple, we deduce that f induces an isomorphism $G \simeq \alpha_p$. □

Remark. In the category of finite flat group schemes, the image of a homomorphism is a scheme theoretic image and thus is closed in the target; in particular, subobjects of a finite flat k-group scheme is a closed k-subgroup.

Example 1.5.24. We say that an elliptic curve E over $\overline{\mathbb{F}}_p$ is *supersingular* if $E[p](\overline{\mathbb{F}}_p)$ is trivial. We assert that every supersingular elliptic curve E over $\overline{\mathbb{F}}_p$ yields a short exact sequence

$$\underline{0} \longrightarrow \alpha_p \longrightarrow E[p] \longrightarrow \alpha_p \longrightarrow \underline{0}.$$

Example 1.1.15 and Theorem 1.5.23 together show that the order of every simple finite flat $\overline{\mathbb{F}}_p$-group is a prime. Since $E[p]$ has order p^2 as noted in Proposition 1.1.16, it is not simple and thus admits a nonzero proper closed $\overline{\mathbb{F}}_p$-subgroup H. Let us consider the exact sequence

$$\underline{0} \longrightarrow H \longrightarrow E[p] \longrightarrow E[p]/H \longrightarrow \underline{0}.$$

Proposition 1.2.13 and Example 1.2.11 together yield a short exact sequence

$$\underline{0} \longrightarrow (E[p]/H)^\vee \longrightarrow E[p] \longrightarrow H^\vee \longrightarrow \underline{0}.$$

We note by Proposition 1.4.7 that $E[p]$ is connected and in turn find by Proposition 1.4.14 that $H, E[p]/H, H^\vee, (E[p]/H)^\vee$ are all connected. Moreover, we see that both H and $E[p]/H$ are simple as they have order p by Theorem 1.1.18. Now Proposition 1.2.8 and Theorem 1.5.23 together imply that both H and $E[p]/H$ are isomorphic to α_p, thereby yielding the desired assertion.

Remark. It turns out that the $\overline{\mathbb{F}}_p$-subgroup $H \simeq \alpha_p$ coincides with $\ker(\varphi_{E[p]})$.

2. p-divisible groups

In this section, we introduce p-divisible groups and discuss their fundamental properties. The primary references for this section are the book of Demazure [**Dem72**] and the article of Tate [**Tat67**]. Throughout this section, we let R denote a noetherian base ring.

2.1. Basic definitions and properties

In this subsection, we define p-divisible groups and describe their basic properties inherited from properties of finite flat group schemes.

Definition 2.1.1. A *p-divisible group of height h* over R is an ind-scheme $G = \varinjlim_{v>0} G_v$ with the following properties:

(i) Each G_v is a finite flat R-group of order p^{vh}.
(ii) Each transition map $i_v : G_v \to G_{v+1}$ fits into a short exact sequence

$$\underline{0} \longrightarrow G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p^v]} G_{v+1}.$$

Example 2.1.2. We present some important examples of p-divisible groups.

(1) The R-group $\underline{0}$ is a p-divisible group of height 0 over R via the identification $\underline{0} \cong \varinjlim \underline{0}$.
(2) The *constant p-divisible group* over R is $\underline{\mathbb{Q}_p/\mathbb{Z}_p} := \varinjlim \underline{\mathbb{Z}/p^v\mathbb{Z}}$ with natural inclusions. It is a p-divisible group of height 1 over R.
(3) The *p-power roots of unity* over R is $\mu_{p^\infty} := \varinjlim \mu_{p^v}$ with natural inclusions. It is a p-divisible group of height 1 over R.
(4) Every abelian scheme $\mathcal{A}$ of dimension g over R gives rises to a p-divisible group $\mathcal{A}[p^\infty] := \varinjlim \mathcal{A}[p^v]$ of height $2g$ over R by Proposition 1.1.16.

Remark. When R has characteristic p, we have a finite flat R-group $\alpha_{p^v} := \operatorname{Spec}(R[t]/t^{p^v})$ for each integer $v \geq 1$ with the natural additive group structure on $\alpha_{p^v}(B) = \{ b \in B : b^{p^v} = 0 \}$ for each R-algebra B. However, the ind-scheme $\varinjlim \alpha_{p^v}$ over R with natural inclusions is not a p-divisible group as $[p]_{\alpha^v}$ vanishes for each $v \geq 1$.

Definition 2.1.3. Let $G = \varinjlim G_v$ and $H = \varinjlim H_v$ be p-divisible groups over R.

(1) An ind-scheme morphism $f = (f_v)$ from G to H is a *homomorphism* if each f_v is an R-group homomorphism.
(2) The *kernel* of a homomorphism $f = (f_v)$ from G to H is $\ker(f) := \varinjlim \ker(f_v)$.

Example 2.1.4. Given a p-divisible group $G = \varinjlim G_v$ over R and an integer n, the *multiplication by n* on G is the homomorphism $[n]_G := ([n]_{G_v})$.

Lemma 2.1.5. Let B be an R-algebra.

(1) Given a p-divisible group $G = \varinjlim G_v$ of height h over R, the base change to B yields a p-divisible group $G_B = \varinjlim (G_v)_B$ of height h over B.
(2) Given a short exact sequence of p-divisible groups over R

$$\underline{0} \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow \underline{0},$$

the base change to B yields a short exact sequence of p-divisible groups

$$\underline{0} \longrightarrow (G')_B \longrightarrow G_B \longrightarrow (G'')_B \longrightarrow \underline{0}.$$

Proof. The assertions are straightforward to verify by Lemma 1.2.1. □

Lemma 2.1.6. Every p-divisible group $G = \varinjlim G_v$ over R yields R-group homomorphisms $i_{v,w} : G_v \to G_{v+w}$ and $j_{v,w} : G_{v+w} \to G_w$ for each $v,\ w \geq 1$ with the following properties:

(i) The map $i_{v,w}$ induces a canonical isomorphism $G_v \cong G_{v+w}[p^v]$.

(ii) There exists a commutative diagram

$$\begin{array}{ccc} G_{v+w} & \xrightarrow{\quad [p^v] \quad} & G_{v+w} \\ & {\scriptstyle j_{v,w}} \searrow \qquad \nearrow {\scriptstyle i_{w,v}} & \\ & G_w & \end{array}$$

(iii) There exists a short exact sequence

$$\underline{0} \longrightarrow G_v \xrightarrow{i_{v,w}} G_{v+w} \xrightarrow{j_{v,w}} G_w \longrightarrow \underline{0}.$$

Proof. Let us write $i_v : G_v \to G_{v+1}$ for the transition map. For each $v,\ w \geq 1$, the map i_{v+w-1} induces a natural isomorphism

$$G_{v+w}[p^v] \cong G_{v+w}[p^{v+w-1}] \cap G_{v+w}[p^v] \cong G_{v+w-1} \cap G_{v+w}[p^v] \cong G_{v+w-1}[p^v].$$

Hence we set $i_{v,w} := i_{v+w-1} \circ \cdots \circ i_v$ and establish property (i) by induction on w. Moreover, as the image of $[p^v]_{G_{v+w}}$ lies in $G_{v+w}[p^w]$ by the fact that $[p^{v+w}]_{G_{v+w}}$ vanishes, property (i) implies that there exists a unique map $j_{v,w} : G_{v+w} \to G_w$ with property (ii).

It remains to verify property (iii). The map $i_{v,w}$ is a closed embedding as easily seen by Proposition 1.1.11. Meanwhile, properties (i) and (ii) together yield an identification

$$\ker(j_{v,w}) = G_{v+w}[p^v] \cong G_v.$$

Hence $j_{v,w}$ gives rise to a closed embedding $G_{v+w}/G_v \hookrightarrow G_w$, which is indeed an isomorphism by Theorem 1.1.18 as both G_{v+w}/G_v and G_w have order p^w. We deduce that $j_{v,w}$ is surjective and in turn establish property (iii). □

Proposition 2.1.7. Given p-divisible groups $G = \varinjlim G_v$ and $H = \varinjlim H_v$ over R, there exists a natural identification

$$\mathrm{Hom}(G, H) \cong \varprojlim \mathrm{Hom}_{R\text{-grp}}(G_v, H_v).$$

Proof. We note that every R-group homomorphism $G_{v+1} \to H_{v+1}$ for each $v \geq 1$ naturally induces an R-group homomorphism $G_v \to H_v$ and in turn obtain an injective map

$$\varprojlim \mathrm{Hom}_{R\text{-grp}}(G_v, H_v) \hookrightarrow \mathrm{Hom}(G, H).$$

Moreover, for every homomorphism $f = (f_v)$ from G to H, we deduce from Lemma 2.1.6 that the image of each f_v lies in H_v as $[p^v]_{G_v}$ vanishes. Hence we establish the desired assertion. □

Proposition 2.1.8. Let $G = \varinjlim G_v$ be a p-divisible group over R.

(1) There exists a canonical identification $G_v \cong \ker([p^v]_G)$ for each $v \geq 1$.

(2) The homomorphism $[p]_G$ is surjective.

Proof. Given an integer $v \geq 1$, we obtain a natural isomorphism $\ker([p^v]_{G_w}) \cong G_v$ for each $w \geq v$ by Lemma 2.1.6 and thus establish statement (1). In addition, we deduce from Lemma 2.1.6 that the map $[p]_{G_{v+1}}$ factors through a surjection $G_{v+1} \twoheadrightarrow G_v$ for each $v \geq 1$ and consequently establish statement (2). □

Remark. Statement (1) shows that the kernel of a homomorphism between two p-divisible groups is not necessarily a p-divisible group. For statement (2), we may define the surjectivity of $[p]_G$ in terms of fpqc sheaves over R.

Proposition 2.1.9. Let $G = \varinjlim G_v$ be a p-divisible group of height h over R.

(1) The ind-scheme $G^\vee := \varinjlim G_v^\vee$ with transition maps induced by $[p]_G$ is a p-divisible group of height h over R.
(2) There exists a canonical isomorphism $G \cong (G^\vee)^\vee$.

Proof. Lemma 2.1.6 yields a commutative diagram

$$\begin{array}{ccccccccccc} & & & & & G_1 & & & & & \\ & & & & \overset{j_{v,1}}{\nearrow} & & \overset{i_{1,v}}{\searrow} & & & & \\ \underline{0} & \longrightarrow & G_v & \xrightarrow{i_v = i_{v,1}} & G_{v+1} & \xrightarrow{\quad [p^v] \quad} & & G_{v+1} & \xrightarrow{j_v = j_{1,v}} & G_v & \longrightarrow \underline{0} \end{array}$$

where the horizontal arrows form an exact sequence. Hence we obtain an exact sequence

$$\underline{0} \longrightarrow G_v^\vee \xrightarrow{j_v^\vee} G_{v+1}^\vee \xrightarrow{[p^v]} G_{v+1}^\vee$$

by Example 1.2.7 and Proposition 1.2.13. Now the desired assertions immediately follow from Theorem 1.2.4. □

Definition 2.1.10. Given a p-divisible group G over R, we refer to the p-divisible group $G^\vee$ in Proposition 2.1.9 as the *Cartier dual* of G.

Remark. Some authors refer to $G^\vee$ as the *Serre dual* of G.

Example 2.1.11. Let us record the Cartier duals of p-divisible groups from Example 2.1.2.

(1) The Cartier dual of $\underline{0}$ is evidently $\underline{0}$ by definition.
(2) We have $(\underline{\mathbb{Q}_p/\mathbb{Z}_p})^\vee \cong \mu_{p^\infty}$ and $\mu_{p^\infty}^\vee \cong \underline{\mathbb{Q}_p/\mathbb{Z}_p}$ by Proposition 1.2.8.
(3) Given an abelian scheme $\mathcal{A}$ over R, we have $\mathcal{A}[p^\infty]^\vee \cong \mathcal{A}^\vee[p^\infty]$ by Proposition 1.2.10 where $\mathcal{A}^\vee$ denotes the dual abelian scheme of $\mathcal{A}$.

Proposition 2.1.12. Assume that R is a henselian local ring with residue field k. Let $G = \varinjlim G_v$ be a p-divisible group over R.

(1) There exists a natural exact sequence of p-divisible groups
$$\underline{0} \longrightarrow G^\circ \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow \underline{0} \tag{2.1}$$
with $G^\circ = \varinjlim G_v^\circ$ and $G^{\text{ét}} = \varinjlim G_v^{\text{ét}}$.
(2) If $R = k$ is a perfect field, the exact sequence (2.1) canonically splits.

Proof. Since the order of G_1 is a power of p, we deduce from Theorem 1.1.18 that the R-groups G_1° and $G_1^{\text{ét}}$ respectively have order p^{h° and $p^{h^{\text{ét}}}$ for some integers h° and $h^{\text{ét}}$. Meanwhile, as Lemma 2.1.6 yields a natural isomorphism $G_{v+1}/G_v \cong G_1$ for each $v \geq 1$, we find $G_{v+1}^\circ/G_v^\circ \cong G_1^\circ$ and $G_{v+1}^{\text{ét}}/G_v^{\text{ét}} \cong G_1^{\text{ét}}$ by Proposition 1.4.14. A simple induction based on Theorem 1.1.18 shows that the R-groups G_v° and $G_v^{\text{ét}}$ respectively have order p^{vh° and $p^{vh^{\text{ét}}}$. In addition, Proposition 1.4.14 yields short exact sequences

$$\underline{0} \longrightarrow G_v^\circ \longrightarrow G_{v+1}^\circ \xrightarrow{[p^v]} G_{v+1}^\circ \quad \text{and} \quad \underline{0} \longrightarrow G_v^{\text{ét}} \longrightarrow G_{v+1}^{\text{ét}} \xrightarrow{[p^v]} G_{v+1}^{\text{ét}}.$$

Therefore $G^\circ = \varinjlim G_v^\circ$ and $G^{\text{ét}} = \varinjlim G_v^{\text{ét}}$ are p-divisible groups over R. Now the desired assertions are evident by Proposition 1.4.13 and Proposition 1.4.15. □

Remark. Proposition 2.1.12 implies an interesting fact that for a p-divisible group $G = \varinjlim G_v$ over a henselian local ring R each G_v being connected or étale is equivalent to G_1 being connected or étale.

Definition 2.1.13. Let $G = \varinjlim G_v$ be a p-divisible group over R.

(1) We say that G is *connected* if each G_v is connected.
(2) We say that G is *étale* if each G_v is étale.
(3) If R is a henselian local ring, we refer to the p-divisible groups G° and $G^{\text{ét}}$ in Proposition 2.1.12 respectively as the *connected part* and the *étale part* of G.

Example 2.1.14. Below are essential examples of étale or connected p-divisible groups.

(1) The constant p-divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ is étale by Proposition 1.3.6.
(2) If R is a henselian local ring with residue field of characteristic p, the p-power roots of unity μ_{p^∞} is connected by Example 1.4.8.

Definition 2.1.15. Assume that $R = k$ is a field of characteristic p. Let $G = \varinjlim G_v$ be a p-divisible group over k and r be a positive integer.

(1) The p^r*-Frobenius twist* of G is $G^{(p^r)} := \varinjlim G_v^{(p^r)}$ with transition maps given by the p^r-Frobenius twists of the transition maps for G.
(2) We define the p^r*-Frobenius* of G to be $\varphi_G^{[r]} := (\varphi_{G_v}^{[r]})$ and the p^r*-Verschiebung* of G to be $\psi_G^{[r]} := (\psi_{G_v}^{[r]})$.
(3) For $r = 1$, we often refer to $\varphi_G := \varphi_G^{[1]}$ and $\psi_G := \psi_G^{[1]}$ respectively as the *Frobenius* and the *Verschiebung* of G.

Proposition 2.1.16. Assume that $R = k$ is a field of characteristic p. Let G be a p-divisible group of height h over k and r be a positive integer.

(1) The ind-scheme $G^{(p^r)}$ is a p-divisible group of height h over k.
(2) The maps $\varphi_G^{[r]}$ and $\psi_G^{[r]}$ are homomorphisms of p-divisible groups.
(3) We have $\psi_G^{[r]} \circ \varphi_G^{[r]} = [p^r]_G$ and $\varphi_G^{[r]} \circ \psi_G^{[r]} = [p^r]_{G^{(p^r)}}$.

Proof. The assertions are direct consequences of Proposition 1.5.8, Lemma 1.5.13, and Proposition 1.5.15. □

Definition 2.1.17. If $R = k$ is a field, for a p-divisible group $G = \varinjlim G_v$ over k we define its *Tate module* to be $T_p(G) := \varprojlim G_v(\overline{k})$ with transition maps induced by $[p]_G$.

Proposition 2.1.18. If $R = k$ is a field, for a p-divisible group $G = \varinjlim G_v$ over k the Tate module $T_p(G)$ is naturally a finite free $\mathbb{Z}_p$-module with a continuous Γ_K-action.

Proof. The assertion is evident as each $G_v(\overline{k})$ is a finite free module over $\mathbb{Z}/p^v\mathbb{Z}$ and carries a canonical continuous Γ_k-action. □

Proposition 2.1.19. If $R = k$ is a perfect field of characteristic not equal to p, there exists an equivalence of categories

$$\{\, p\text{-divisible groups over } k \,\} \xrightarrow{\sim} \{\, \text{finite free } \mathbb{Z}_p\text{-modules with a continuous } \Gamma_k\text{-action} \,\}$$

which sends each p-divisible group G over k to $T_p(G)$.

Proof. Let $G = \varinjlim G_v$ be a p-divisible group over k. Since all finite flat k-groups of p-power order are étale by Theorem 1.3.9, it is not hard to deduce from Proposition 1.3.4 that the functor is fully faithful. Moreover, given a finite free $\mathbb{Z}_p$-module M with a continuous Γ_k-action, Proposition 1.3.4 yields a finite étale k-group G_v with $G_v(\overline{k}) = M/(p^v)$ for each integer $v \geq 1$ and in turn provides a p-divisible group $G = \varinjlim G_v$ with $T_p(G) = M$. Therefore we deduce that the functor is an equivalence as desired. □

2.2. Serre-Tate equivalence for connected p-divisible groups

In this subsection, we introduce formal group laws and explore their relations to p-divisible groups. Throughout this subsection, we assume that R is a complete reduced noetherian local ring with residue field k of characteristic p. We often denote the ring $R[[t_1, \cdots, t_d]]$ of power series in d variables by $\mathscr{A}_d$, or simply by $\mathscr{A}$ if the context clearly specifies d. We work with the canonical identifications $\mathscr{A}_d \widehat{\otimes}_R \mathscr{A}_d \cong R[[T, U]]$ and $\mathscr{A}_d \widehat{\otimes}_R \mathscr{A}_d \widehat{\otimes}_R \mathscr{A}_d \cong R[[T, U, V]]$, where we write $T := (t_1, \cdots, t_d)$, $U := (u_1, \cdots, u_d)$, and $V := (v_1, \cdots, v_d)$.

Lemma 2.2.1. An R-algebra homomorphism $f : R[[t_1, \cdots, t_n]] \to R[[u_1, \cdots, u_m]]$ is continuous if and only if each $f(t_i)$ lies in the ideal $\mathscr{I} := (u_1, \cdots, u_m)$.

Proof. The map f is continuous if and only if there exists an integer v with $f(t_i^v) \in \mathscr{I}$ for each $i = 1, \cdots, n$. Hence the assertion follows from our assumption that R is reduced. □

Definition 2.2.2. A *formal group law of dimension* d over R is a continuous R-algebra homomorphism $\mu : \mathscr{A}_d \to \mathscr{A}_d \widehat{\otimes}_R \mathscr{A}_d$ such that $\Phi(T, U) := (\mu(t_i))$ satisfies the following axioms:

(a) associativity axiom $\Phi(T, \Phi(U, V)) = \Phi(\Phi(T, U), V)$,
(b) unit section axiom $\Phi(T, 0) = T = \Phi(0, T)$,
(c) commutativity axiom $\Phi(T, U) = \Phi(U, T)$.

Example 2.2.3. We present two primary examples of one-dimensional formal group laws.

(1) The *additive formal group law* over R is the continuous R-algebra homomorphism $\mu_{\widehat{\mathbb{G}_a}} : R[[t]] \to R[[t, u]]$ with $\mu_{\widehat{\mathbb{G}_a}}(t) = t + u$.
(2) The *multiplicative formal group law* over R is the continuous R-algebra homomorphism $\mu_{\widehat{\mathbb{G}_m}} : R[[t]] \to R[[t, u]]$ with $\mu_{\widehat{\mathbb{G}_m}}(t) = (1 + t)(1 + u) - 1$.

Lemma 2.2.4. Let $\mu : \mathscr{A} \to \mathscr{A} \widehat{\otimes}_R \mathscr{A}$ be a formal group law of dimension d over R represented by $\Phi(T, U) := (\mu(t_i))$. We have a d-tuple $\Xi(T) = (\Xi_i(T))$ of power series in d variables with

$$\Phi(T, \Xi(T)) = 0 = \Phi(\Xi(T), T).$$

Proof. By the commutativity axiom for μ, it suffices to construct a d-tuple $\Xi(T)$ with $\Phi(T, \Xi(T)) = 0$. Let us consider the ideal $\mathscr{I} := (t_1, \cdots, t_d)$ of $\mathscr{A}$. We have a natural identification $\mathscr{I} \widehat{\otimes} \mathscr{I} \cong (t_1, \cdots, t_d, u_1, \cdots, u_d)$. For each R-module M, we regard $M^{\times d}$ as the set of d-tuples whose entries all lie in M. We wish to present the desired d-tuple as a limit $\Xi(T) = \lim_{j \to \infty} P_j(T)$ where each $P_j(T)$ is a d-tuple of polynomials with

$$P_j(T) \in P_{j-1}(T) + (\mathscr{I}^j)^{\times d} \quad \text{and} \quad \Phi(P_j(T), T) \in (\mathscr{I}^{j+1})^{\times d}.$$

The unit section axiom for μ yields the relation

$$\Phi(T, U) \in T + U + ((\mathscr{I} \widehat{\otimes} \mathscr{I})^2)^{\times d}. \tag{2.2}$$

Let us set $P_1(T) := -T$ and inductively construct $P_j(T)$ for each $j > 1$. By the relation $\Phi(P_{j-1}(T), T) \in (\mathscr{I}^j)^{\times d}$, there exists a d-tuple $\Delta_j(T) \in (\mathscr{I}^j)^{\times d}$ with

$$\Delta_j(T) \in -\Phi(P_{j-1}(T), T) + (\mathscr{I}^{j+1})^{\times d}. \tag{2.3}$$

For $P_j(T) := P_{j-1}(T) + \Delta_j(T)$, we have $P_j(T) \in P_{j-1}(T) + (\mathscr{I}^j)^{\times d}$ and find

$$\Phi(P_j(T), T) = \Phi(P_{j-1}(T) + \Delta_j(T), T) \in \Phi(P_{j-1}(T), T) + \Delta_j(T) + (\mathscr{I}^{j+1})^{\times d} = (\mathscr{I}^{j+1})^{\times d}$$

by the relations (2.2) and (2.3). Therefore we obtain a desired d-tuple $\Xi(T)$. □

Remark. Lemma 2.2.4 shows that the inverse axiom is automatic for formal group laws.

LEMMA 2.2.5. Let $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ be a formal group law of dimension d over R.

(1) The formal group law μ yields commutative diagrams

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\mu} & \mathscr{A}\widehat{\otimes}_R\mathscr{A} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu\widehat{\otimes}\mathrm{id}} \\ \mathscr{A}\widehat{\otimes}_R\mathscr{A} & \xrightarrow{\mathrm{id}\widehat{\otimes}\mu} & \mathscr{A}\widehat{\otimes}_R\mathscr{A}\widehat{\otimes}_R\mathscr{A} \end{array} \qquad \begin{array}{ccc} \mathscr{A}\widehat{\otimes}_R\mathscr{A} & \xrightarrow{x\widehat{\otimes}y\mapsto y\widehat{\otimes}x} & \mathscr{A}\widehat{\otimes}_R\mathscr{A} \\ {\scriptstyle \mu}\nwarrow & & \nearrow{\scriptstyle \mu} \\ & \mathscr{A} & \end{array}$$

(2) The R-algebra map $\epsilon : \mathscr{A} \to R$ with $\epsilon(t_i) = 0$ fits into commutative diagrams

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{\mathrm{id}} & \mathscr{A} & \xrightarrow{\sim} & \mathscr{A}\widehat{\otimes}_R R \\ & {\scriptstyle \mu}\searrow & & \nearrow{\scriptstyle \mathrm{id}\widehat{\otimes}\epsilon} & \\ & & \mathscr{A}\widehat{\otimes}_R\mathscr{A} & & \end{array} \qquad \begin{array}{ccccc} \mathscr{A} & \xrightarrow{\mathrm{id}} & \mathscr{A} & \xrightarrow{\sim} & R\widehat{\otimes}_R\mathscr{A} \\ & {\scriptstyle \mu}\searrow & & \nearrow{\scriptstyle \epsilon\widehat{\otimes}\mathrm{id}} & \\ & & \mathscr{A}\widehat{\otimes}_R\mathscr{A} & & \end{array}$$

(3) There exists an R-algebra map $\iota : \mathscr{A} \to \mathscr{A}$ that fits into a commutative diagram

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\mu} & \mathscr{A}\widehat{\otimes}_R\mathscr{A} \\ \downarrow{\scriptstyle \epsilon} & & {\scriptstyle \mathrm{id}\widehat{\otimes}\iota}\downarrow\downarrow{\scriptstyle \iota\widehat{\otimes}\mathrm{id}} \\ R & \longrightarrow & \mathscr{A} \end{array}$$

PROOF. Statements (1) and (2) are evident by the axioms for μ. Statement (3) is a reformulation of Lemma 2.2.4. $\square$

Remark. We can extend the notion of R-groups to define formal R-groups as group objects in the category of formal R-schemes. Lemma 2.2.5 shows that every formal group law μ of dimension d over R corresponds to a unique a formal R-group $\mathscr{G}_\mu = \mathrm{Spf}(\mathscr{A})$ with comultiplication μ, counit ϵ, and coinverse ι.

Definition 2.2.6. Let μ and ν be formal group laws over R.

(1) A *homomorphism* from μ to ν is a continuous R-algebra map $\theta : \mathscr{A}_{d'} \to \mathscr{A}_d$ with a commutative diagram

$$\begin{array}{ccc} \mathscr{A}_{d'} & \xrightarrow{\nu} & \mathscr{A}_{d'}\widehat{\otimes}_R\mathscr{A}_{d'} \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta\widehat{\otimes}\theta} \\ \mathscr{A}_d & \xrightarrow{\mu} & \mathscr{A}_d\widehat{\otimes}_R\mathscr{A}_d \end{array}$$

where d and d' respectively denotes the dimensions of μ and ν.

(2) A homomorphism $\theta : \mathscr{A}_{d'} \to \mathscr{A}_d$ from μ and ν is *finite flat* if $\mathscr{A}_d$ becomes a free module of finite rank over $\mathscr{A}_{d'}$ via θ.

Remark. The map θ goes from the power series ring for ν to the power series ring for μ so that it corresponds to a formal R-group homomorphism $\mathscr{G}_\mu \to \mathscr{G}_\nu$. If we consider the tuples $\Phi(T, U) := (\mu(t_i))$, $\Psi(T, U) := (\nu(t_j))$, and $\Xi(T) := (\theta(t_j))$, the commutative diagram for θ is equivalent to the identity $\Psi(\Xi(T), \Xi(U)) = \Xi(\Phi(T, T))$.

Example 2.2.7. Let μ be a formal group law of dimension d over R. For each integer $n \geq 1$, the *multiplication by* n on μ is the homomorphism $[n]_\mu : \mathscr{A} \to \mathscr{A}$ inductively defined by the relations $[1]_\mu = \mathrm{id}_{\mathscr{A}}$ and $[n]_\mu = ([n-1]_\mu\widehat{\otimes}\mathrm{id}) \circ \mu$.

Remark. The map $[n]_\mu$ induces the multiplication by n on the formal R-group $\mathscr{G}_\mu$.

Definition 2.2.8. Let $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ be a formal group law of dimension d over R.

(1) We refer to the ideal $\mathscr{I} := (t_1, \cdots, t_d)$ in $\mathscr{A}$ as the *augmentation ideal* of μ.
(2) We say that μ is *p-divisible* if the homomorphism $[p]_\mu : \mathscr{A} \to \mathscr{A}$ is finite flat.

Remark. The ideal $\mathscr{I}$ is the kernel of the counit $\epsilon : \mathscr{A} \to R$ for the formal R-group $\mathscr{G}_\mu$. Hence our definition here is comparable to the definition of augmentation ideal for affine R-groups.

Example 2.2.9. Let us consider the formal group laws introduced in Example 2.2.3.

(1) The additive formal group law $\mu_{\widehat{\mathbb{G}_a}}$ is not p-divisible; indeed, $[p]_{\mu_{\widehat{\mathbb{G}_a}}}$ satisfies the identity $[p]_{\mu_{\widehat{\mathbb{G}_a}}}(t) = pt$ and thus is not finite flat for inducing a zero map on $\mathscr{A} \otimes_R k$.
(2) The multiplicative formal group law $\mu_{\widehat{\mathbb{G}_m}}$ is p-divisible; indeed, $[p]_{\mu_{\widehat{\mathbb{G}_m}}}$ satisfies the identity $[p]_{\mu_{\widehat{\mathbb{G}_m}}}(t) = (1+t)^p - 1$ and thus is finite flat.

Proposition 2.2.10. Let $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ be a p-divisible formal group law of dimension d over R with augmentation ideal $\mathscr{I}$. We write $A_v := \mathscr{A}/[p^v]_\mu(\mathscr{I})$ for each $v \geq 1$.

(1) Each $\mu[p^v] := \mathrm{Spec}\,(A_v)$ is naturally a connected finite flat R-group.
(2) The ind-scheme $\mu[p^\infty] := \varinjlim \mu[p^v]$ is a connected p-divisible group over R.

Proof. Let us take ϵ and ι as in Lemma 2.2.5. For each $v \geq 1$, we have

$$A_v = \mathscr{A}/[p^v]_\mu(\mathscr{I}) \cong \mathscr{A}/\mathscr{I} \otimes_{\mathscr{A},[p^v]_\mu} \mathscr{A} \cong R \otimes_{\mathscr{A},[p^v]_\mu} \mathscr{A}$$

and thus find that $\mu[p^v] = \mathrm{Spec}\,(A_v)$ is naturally an R-group with comultiplication $1 \otimes \mu$, counit $1 \otimes \epsilon$, and coinverse $1 \otimes \iota$. If we take a basis of $\mathscr{A}$ over $[p]_\mu(\mathscr{A})$ given by $f_1, \cdots, f_r \in \mathscr{A}$, a simple induction yields a basis of $\mathscr{A}$ over $[p^v]_\mu(\mathscr{A})$ for each $v \geq 1$ given by elements of the form $[p^{v-1}]_\mu(f_{i_{v-1}}) \cdots [p]_\mu(f_{i_1}) f_{i_0}$ with $(i_0, \cdots, i_{v-1}) \in (\mathbb{Z}/r\mathbb{Z})^v$ and consequently implies that $\mu[p^v]$ is finite flat of order r^v over R. Moreover, since R is a local ring, both $\mathscr{A}$ and $A_v = \mathscr{A}/[p^v]_\mu(\mathscr{I})$ are local rings as well. We deduce that $\mu[p^v]$ is connected and in turn establish statement (1).

Let us now consider statement (2). Lemma 1.4.3 and Proposition 1.5.20 together imply that $\mu[p]$ has order p^h for some integer h. Therefore our discussion in the previous paragraph shows that each $\mu[p^v]$ has order p^{vh}. Furthermore, the R-algebra homomorphism

$$A_v = \mathscr{A}/[p^v]_\mu(\mathscr{I}) \longrightarrow [p]_\mu(\mathscr{A})/[p^{v+1}]_\mu(\mathscr{I})$$

induced by $[p]_\mu$ is an isomorphism for being a surjective map between two free R-algebras of the same rank. Hence we obtain a surjective ring homomorphism

$$A_{v+1} = \mathscr{A}/[p^{v+1}]_\mu(\mathscr{I}) \twoheadrightarrow [p]_\mu(\mathscr{A})/[p^{v+1}]_\mu(\mathscr{I}) \simeq A_v,$$

which induces an embedding $i_v : \mu[p^v] \hookrightarrow \mu[p^{v+1}]$. Since it is evident by construction that i_v identifies $\mu[p^v]$ with the kernel of $[p^v]$ on $\mu[p^{v+1}]$, we conclude that $\mu[p^\infty] := \varinjlim \mu[p^v]$ is a connected p-divisible group of height h over R, thereby completing the proof. □

Remark. We can alternatively deduce statement (2) from statement (1) by the identifidcation $\mu[p^v] \cong \mathscr{G}_\mu[p^v]$ for each $v \geq 1$.

Definition 2.2.11. Given a p-divisible formal group law μ over R, we define its *associated connected p-divisible group* over R to be $\mu[p^\infty]$ as constructed in Proposition 2.2.10.

Example 2.2.12. The multiplicative formal group law $\mu_{\widehat{\mathbb{G}_m}}$ is p-divisible as explained in Example 2.2.9. For each $v \geq 1$, we have $[p^v]_{\mu_{\widehat{\mathbb{G}_m}}}(t) = (1+t)^{p^v} - 1$ and thus find $\mu_{\widehat{\mathbb{G}_m}}[p^v] \cong \mu_{p^v}$ by Example 1.1.7. Hence we obtain a natural identification $\mu_{\widehat{\mathbb{G}_m}}[p^\infty] \cong \mu_{p^\infty}$.

Our main objective for this subsection is to prove a theorem of Serre and Tate that the association described in Proposition 2.2.10 defines an equivalence between the category of p-divisible formal group laws and the category of connected p-divisible groups.

LEMMA 2.2.13. Let $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ be a formal group law of dimension d over R with augmentation ideal $\mathscr{I}$. For each integer $n \geq 1$, we have

$$[n]_\mu(t_i) \in nt_i + \mathscr{I}^2.$$

PROOF. Let us take d-tuples $\Phi(T, U) := (\mu(t_i))$ and $\Xi_n(T) := ([n]_\mu(t_i))$ for each $n \geq 1$. Given an R-module M, we regard $M^{\times d}$ as the set of d-tuples whose entries all lie in M. Under the natural identification $\mathscr{I}\widehat{\otimes}\mathscr{I} \cong (t_1, \cdots, t_d, u_1, \cdots, u_d)$, we find

$$\Phi(T, U) \in T + U + ((\mathscr{I}\widehat{\otimes}\mathscr{I})^2)^{\times d}.$$

by the unit section axiom for μ. Hence the identity $[n]_\mu = ([n-1]_\mu\widehat{\otimes}\mathrm{id}) \circ \mu$ yield the relation

$$\Xi_n(T) = \Phi(\Xi_{n-1}(T), T) \in \Xi_{n-1}(T) + T + (\mathscr{I}^2)^{\times d}.$$

Since we have $\Xi_1(T) = T$ by definition, we proceed by induction to find $\Xi_n(T) \in nT + (\mathscr{I}^2)^{\times d}$ for each $n \geq 1$, thereby completing the proof. $\square$

Remark. The proof of Theorem 1.3.9 yields an analogous relation for finite flat R-groups.

LEMMA 2.2.14. Given a p-divisible formal group law $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ of dimension d over R with augmentation ideal $\mathscr{I}$, there exists a natural topological R-algebra isomorphism

$$\mathscr{A} \cong \varprojlim A_v$$

where we write $A_v := \mathscr{A}/[p^v]_\mu(\mathscr{I})$ for each $v \geq 1$.

PROOF. Since R is a local ring, $\mathscr{A}$ and A_v are also local rings for each $v \geq 1$. Moreover, each A_v is a free R-algebra of finite rank as noted in Proposition 2.2.10. Let us write $\mathfrak{m}$ for the maximal ideal of R and $\mathfrak{M} := \mathfrak{m}\mathscr{A} + \mathscr{I}$ for the maximal ideal of $\mathscr{A}$. We have $[p]_\mu(\mathscr{I}) \subseteq p\mathscr{I} + \mathscr{I}^2 \subseteq \mathfrak{M}\mathscr{I}$ by Lemma 2.2.13 and thus find $[p^v]_\mu(\mathscr{I}) \subseteq \mathfrak{M}^v\mathscr{I}$ for each $v \geq 1$. Hence for each $i,\ v \geq 1$ we have $[p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A} \subseteq \mathfrak{M}^w$ for some $w \geq 1$. Meanwhile, for each $i,\ v \geq 1$ we find $\mathfrak{M}^{w'} \subseteq [p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A}$ for some $w' \geq 1$ as $\mathscr{A}/([p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A}) = A_v/\mathfrak{m}^iA_v$ is local artinian. Now we obtain a topological R-algebra isomorphism

$$\mathscr{A} \cong \varprojlim_{w} \mathscr{A}/\mathfrak{M}^w \cong \varprojlim_{i,v} \mathscr{A}/([p^v]_\mu(\mathscr{I}) + \mathfrak{m}^i\mathscr{A}) \cong \varprojlim_{v,i} A_v/\mathfrak{m}^iA_v \cong \varprojlim_{v} A_v$$

where the last identification comes from an observation that each A_v is $\mathfrak{m}$-adically complete by a general fact stated in the Stacks project [**Sta**, Tag 031B]. $\square$

LEMMA 2.2.15. Given p-divisible formal group laws μ and ν over R, there exists a natural identification

$$\mathrm{Hom}(\mu, \nu) \cong \mathrm{Hom}(\mu[p^\infty], \nu[p^\infty]).$$

PROOF. Let us write d and d' respectively for the dimensions of μ and ν. In addition, we set $A_v := \mathscr{A}_d/[p^v]_\mu(\mathscr{I}_\mu)$ and $B_v := \mathscr{A}_{d'}/[p^v]_\nu(\mathscr{I}_\nu)$ for each $v \geq 1$, where $\mathscr{I}_\mu$ and $\mathscr{I}_\nu$ respectively denote the augmentation ideals of μ and ν. Proposition 2.2.10 shows that $\mu[p^v] := \mathrm{Spec}\,(A_v)$ and $\nu[p^v] := \mathrm{Spec}\,(B_v)$ are connected finite flat R-groups. Since we have $\mathscr{A}_d \cong \varprojlim A_v$ and $\mathscr{A}_{d'} \cong \varprojlim B_v$ by Lemma 2.2.14, we apply Proposition 2.1.7 to obtain a natural identification

$$\mathrm{Hom}(\mu, \nu) \cong \varprojlim \mathrm{Hom}_{\nu_v,\mu_v}(B_v, A_v) \cong \varprojlim \mathrm{Hom}_{R\text{-grp}}(\mu[p^v], \nu[p^v]) \cong \mathrm{Hom}(\mu[p^\infty], \nu[p^\infty])$$

where $\mathrm{Hom}_{\nu_v,\mu_v}(B_v, A_v)$ denotes the set of R-algebra maps $B_v \to A_v$ compatible with the comultiplications μ_v on $\mu[p^v]$ and ν_v on $\nu[p^v]$. $\square$

Proposition 2.2.16. Let $G = \varinjlim G_v$ be a connected p-divisible group over R.

(1) There exists a topological k-algebra isomorphism

$$\varprojlim (A_v \otimes_R k) \simeq k[[t_1, \cdots, t_d]] \quad \text{for some } d \geq 0$$

where A_v denotes the affine ring of G_v.

(2) The special fiber $\overline{G} := G \times_R k$ is a p-divisible group over k such that $\ker(\varphi_{\overline{G}})$ is a finite flat k-group of order p^d.

Proof. It is evident by Lemma 2.1.5 that $\overline{G}$ is a p-divisible group over k. Let us write $\overline{G}_v := G_v \times_R k$ and $H_v := \ker(\varphi_{\overline{G}}^{[v]})$ for each $v \geq 1$. Proposition 2.1.8 and Proposition 2.1.16 together imply that each H_v is a closed k-subgroup of $\overline{G}[p^v] \cong \overline{G}_v$. Moreover, each $\overline{G}_v$ is connected by Lemma 1.4.3 and thus is a k-subgroup of $\ker(\varphi_{\overline{G}}^{[w]}) = H_w$ for some $w \geq 1$ by Proposition 1.5.17. Now we write $H_v = \operatorname{Spec}(B_v)$ for each $v \geq 1$ and obtain a topological k-algebra isomorphism

$$\varprojlim A_v \otimes_R k \simeq \varprojlim B_v. \tag{2.4}$$

We denote the augmentation ideal of H_v by J_v and set $J := \varprojlim J_v$. Since each H_v is connected, as easily seen by Proposition 1.5.17, its affine ring B_v is a local k-algebra with maximal ideal J_v by Lemma 1.5.16. In addition, we have $H_1 \cong \ker(\varphi_{H_v})$ by Lemma 1.5.7 and thus apply Proposition 1.5.19 to obtain an isomorphism $B_1 \cong B_v/J_v^{(p)}$ where $J_v^{(p)}$ denotes the ideal generated by the p-th powers of elements in J_v. Now we find $J_1 \cong J_v/J_v^{(p)}$ and in turn get an identification $J_1/J_1^2 \cong J_v/J_v^2$. Let us take $b_1, \cdots, b_d \in J$ whose images in J_1/J_1^2 form a k-basis. Nakayama's lemma implies that J_v admits generators given by the images of $b_1, \cdots, b_d$ and consequently yields a surjective k-algebra homomorphism $k[t_1, \cdots, t_d] \twoheadrightarrow B_v$ which sends each t_i to the image of b_i in B_v. Furthermore, as $\varphi_{H_v}^{[v]}$ vanishes by Lemma 1.5.7, this map induces a surjective k-algebra homomorphism

$$\lambda_v : k[t_1, \cdots, t_d]/(t_1^{p^v}, \cdots, t_d^{p^v}) \twoheadrightarrow B_v$$

by Proposition 1.5.19. Hence we obtain a continuous k-algebra homomorphism

$$\lambda : k[[t_1, \cdots, t_d]] \twoheadrightarrow \varprojlim B_v$$

via the identification $k[[t_1, \cdots, t_d]] \cong \varprojlim k[t_1, \cdots, t_d]/(t_1^{p^v}, \cdots, t_d^{p^v})$.

In light of the isomorphism (2.4), we wish to show that λ is a topological isomorphism. We only need to prove that each λ_v is an isomorphism. Since each λ_v is surjective by construction, it suffices to verify that its source and target have equal k-dimensions; in other words, it is enough to show that each B_v has k-dimension p^{dv}, or equivalently that each H_v has order p^{dv}.

For $v = 1$, the assertion follows from Proposition 1.5.19. Let us henceforth assume the inequality $v > 1$ and proceed by induction. Proposition 2.1.16 shows that $\overline{G}^{(p)}$ is a p-divisible group over k with $\varphi_{\overline{G}} \circ \psi_{\overline{G}} = [p]_{\overline{G}^{(p)}}$. Moreover, as $[p]_{\overline{G}^{(p)}}$ is surjective by Proposition 2.1.8, the map $\varphi_{\overline{G}}$ is surjective and thus maps $H_v = \ker(\varphi_{\overline{G}}^{[v]})$ surjectively onto $\ker(\varphi_{\overline{G}^{(p)}}^{[v-1]}) \cong H_{v-1}^{(p)}$. We deduce that there exists a short exact sequence

$$\underline{0} \longrightarrow H_1 \longrightarrow H_v \longrightarrow H_{v-1}^{(p)} \longrightarrow \underline{0}.$$

Now the desired assertion follows from Theorem 1.1.18 and the fact that the order of $H_{v-1}^{(p)}$ is the same as the order of H_{v-1}. ☐

Remark. The ind-scheme $H = \varinjlim H_v$ is not necessarily a p-divisible group.

Lemma 2.2.17. Given an R-algebra B, its ideal J with $J \otimes_R k = 0$ is trivial if for each maximal ideal $\mathfrak{n}$ of B the $B_{\mathfrak{n}}$-module $J_{\mathfrak{n}}$ admits a finite set of generators.

Proof. Let us write $\mathfrak{m}$ for the maximal ideal of R. For each maximal ideal $\mathfrak{n}$ of B, we have $J_{\mathfrak{n}} = \mathfrak{m} J_{\mathfrak{n}} \subseteq \mathfrak{n} J_{\mathfrak{n}}$ and thus deduce from Nakayama's lemma that $J_{\mathfrak{n}}$ is trivial. □

Lemma 2.2.18. Let $G = \varinjlim G_v$ be a p-divisible group over R with $G_v = \mathrm{Spec}\,(A_v)$.

(1) G gives rise to a flat R-algebra $\varprojlim A_v$.
(2) If an R-algebra B admits a k-algebra isomorphism

$$\overline{\theta} : (B \otimes_R k)[[t_1, \cdots, t_d]] \xrightarrow{\sim} \varprojlim (A_v \otimes_R k) \quad \text{for some } d \geq 0,$$

there exists an R-algebra surjection $\theta : B[[t_1, \cdots, t_d]] \twoheadrightarrow \varprojlim A_v$ which lifts $\overline{\theta}$.

Proof. Since each $i_v : G_v \to G_{v+1}$ is a closed embedding by Proposition 1.1.11, the induced map $\pi_v : A_{v+1} \to A_v$ is surjective. Hence statement (1) follows from a general fact stated in the Stacks project [**Sta**, Tag 0912]. It remains to establish statement (2).

We assert that each $\overline{\theta}_v : (B \otimes_R k)[[t_1, \cdots, t_d]] \twoheadrightarrow A_v \otimes_R k$ lifts to an R-algebra homomorphism $\theta_v : B[[t_1, \cdots, t_d]] \to A_v$ with a commutative diagram

$$\begin{array}{ccccc} B[[t_1, \cdots, t_d]] & \xrightarrow{\theta_{v+1}} & A_{v+1} & \longrightarrow & A_{v+1} \otimes_R k \\ & \searrow^{\theta_v} & \downarrow\!\!\downarrow \pi_v & & \downarrow\!\!\downarrow \pi_v \otimes \mathrm{id} \\ & & A_v & \longrightarrow & A_v \otimes_R k \end{array}$$

We take θ_1 to be an arbitrary lift of $\overline{\theta}_1$ and proceed by induction on v. Let us write $\mathfrak{m}$ for the maximal ideal of R and choose $a_1, \cdots, a_d \in A_{v+1}$ which lift $\overline{\theta}_{v+1}(t_1), \cdots, \overline{\theta}_{v+1}(t_d)$. We observe that $\pi_v(a_1), \cdots, \pi_v(a_d)$ lift $\overline{\theta}_v(t_1), \cdots, \overline{\theta}_v(t_d)$ and in turn find $\theta_v(t_i) - \pi_v(a_i) \in \mathfrak{m} A_v$. Since π_v is surjective, we may choose $b_1, \cdots, b_d \in \mathfrak{m} A_{v+1}$ with $\pi_v(b_i) = \theta_v(t_i) - \pi_v(a_i)$ and deduce that $\overline{\theta}_{v+1}$ lifts to a map $\theta_{v+1} : B[[t_1, \cdots, t_d]] \to A_{v+1}$ with $\theta_{v+1}(t_i) = a_i + b_i$ as desired.

Now we have an R-algebra homomorphism $\theta : B[[t_1, \cdots, t_d]] \to \varprojlim A_v$ which lifts $\overline{\theta}$. We find $\mathrm{coker}(\theta) \otimes_R k = \mathrm{coker}(\overline{\theta}) = 0$ and also observe that $\mathrm{coker}(\theta)$ admits a generator over $\varprojlim A_v$ given by the image of 1. Therefore Lemma 2.2.17 implies that θ is surjective. □

Lemma 2.2.19. Every connected p-divisible group $G = \varinjlim G_v$ over R with $G_v = \mathrm{Spec}\,(A_v)$ yields a formal group law $\mu : \mathscr{A} \to \mathscr{A} \widehat{\otimes}_R \mathscr{A}$ via a topological R-algebra isomorphism

$$\mathscr{A} = R[[t_1, \cdots, t_d]] \simeq \varprojlim A_v \quad \text{for some } d \geq 0.$$

Proof. Proposition 2.2.16 and Lemma 2.2.18 yield a surjective R-algebra homomorphism $\theta : \mathscr{A} \twoheadrightarrow \varprojlim A_v$ which lifts a topological isomorphism $\overline{\theta} : k[[t_1, \cdots, t_d]] \xrightarrow{\sim} \varprojlim (A_v \otimes_R k)$. In addtion, we have $\ker(\theta) \otimes_R k = \ker(\overline{\theta}) = 0$ by Lemma 2.2.18 and a general fact stated in the Stacks project [**Sta**, Tag 00HL]. Since $\mathscr{A}$ is a notherian local ring, we find $\ker(\theta) = 0$ by Lemma 2.2.17 and in turn deduce that θ is an isomorphism.

The map θ is continuous as the kernel of each $\theta_v : \mathscr{A} \to A_v$ is open by the fact that the R-algebra A_v is of finite length. Moreover, with $\overline{\theta}$ being a topological isomorphism, we observe that every power of the ideal $\mathscr{I} := (t_1, \cdots, t_d)$ contains an open set in its image under θ and consequently find that θ is open. Therefore θ is a topological R-algebra isomorphism.

Let us denote the comultiplication of each G_v by μ_v. Via the isomorphism θ we may identify $\varprojlim \mu_v$ with a continuous R-algebra homomorphism $\mu : \mathscr{A} \to \mathscr{A} \widehat{\otimes}_R \mathscr{A}$. It is evident by the axioms for each comultiplication μ_v that μ is indeed a formal group law over R. □

THEOREM 2.2.20 (Serre-Tate). There exists an equivalence of categories

$$\{\, p\text{-divisible formal group laws over } R \,\} \xrightarrow{\sim} \{\, \text{connected } p\text{-divisible groups over } R \,\}$$

which maps each p-divisible formal group law μ over R to $\mu[p^\infty]$.

PROOF. Since Lemma 2.2.15 shows that the functor is fully faithful, we only need to prove that the functor is essentially surjective. Let $G = \varinjlim G_v$ be an arbitrary connected p-divisible group of height h over R with $G_v = \mathrm{Spec}\,(A_v)$. Lemma 2.2.19 yields a formal group law $\mu : \mathscr{A} \to \mathscr{A}\widehat{\otimes}_R\mathscr{A}$ induced by G via a topological R-algebra isomorphism

$$\mathscr{A} = R[[t_1, \cdots, t_d]] \simeq \varprojlim A_v \qquad \text{for some } d \geq 0.$$

We wish to show that μ is p-divisible with $\mu[p^\infty] \simeq G$.

We denote the agumentation ideal of $\mathscr{A}$ by $\mathscr{I}$. For each $v \geq 1$, we have $G_v \cong \ker([p^v]_G)$ by Proposition 2.1.8 and thus find $A_v \simeq \mathscr{A}/[p^v]_\mu(\mathscr{I})$. Let us write $r := p^h$ and choose $f_1, \cdots, f_r \in \mathscr{A}$ whose images in $A_1 \simeq \mathscr{A}/[p]_\mu(\mathscr{I})$ form a basis over R.

For every $g \in \mathscr{A}$, a simple induction yields a sequence $(g_{i,j})$ for each $i = 1, \cdots, r$ with

$$g_{i,j} \in g_{i,j-1} + \mathscr{I}^{j-1} \qquad \text{and} \qquad g \in \sum_{i=1}^{r} [p]_\mu(g_{i,j}) f_i + [p]_\mu(\mathscr{I})^j.$$

Since we have $[p]_\mu(\mathscr{I}) \subseteq \mathscr{I}$ by Lemma 2.2.1, we set $g_i := \lim\limits_{j\to\infty} g_{i,j}$ and find $g = \sum\limits_{i=1}^{r} [p]_\mu(g_i) f_i$. Therefore we deduce that $f_1, \cdots, f_r$ generate $\mathscr{A}$ over $[p]_\mu(\mathscr{A})$.

Meanwhile, each $[p]_{G_v}$ factors through a surjective R-group homomorphism $G_{v+1} \twoheadrightarrow G_v$ by Lemma 2.1.6 and in turn induces a faithfully flat R-algebra homomorphism

$$\eta_v : A_v \simeq \mathscr{A}/[p^v]_\mu(\mathscr{I}) \longrightarrow \mathscr{A}/[p^{v+1}]_\mu(\mathscr{I}) \simeq A_{v+1}$$

by a standard fact stated in the Stacks project [**Sta**, Tag 00HQ]. As we know that each A_v is a free local R-algebra of rank p^{vh}, we see that A_{v+1} is free over A_v of rank $r = p^h$ by some general facts stated in the Stacks project [**Sta**, Tag 08WP and Tag 00NZ]. Hence the images of $f_1, \cdots, f_r$ in $A_{v+1} \simeq \mathscr{A}/[p^{v+1}]_\mu(\mathscr{I})$ form a basis over $A_v \simeq \mathscr{A}/[p^v]_\mu(\mathscr{I})$.

Let us now consider a relation $\sum\limits_{i=1}^{r} [p]_\mu(h_i) f_i = 0$ with $h_1, \cdots, h_r \in \mathscr{A}$. For each $v \geq 1$, we consider this relation in $A_{v+1} \simeq \mathscr{A}/[p^{v+1}]_\mu(\mathscr{I})$ and find $[p]_\mu(h_1), \cdots, [p]_\mu(h_r) \in [p^v]_\mu(\mathscr{I})$. Since we have $[p^v]_\mu(\mathscr{I}) \subseteq \mathscr{I}^v$ for each $v \geq 1$ as easily seen by Lemma 2.2.1, we deduce that $[p]_\mu(h_1), \cdots, [p]_\mu(h_r)$ must all be zero. Therefore we find that $f_1, \cdots, f_r$ form a basis of $\mathscr{A}$ over $[p]_\mu(\mathscr{A})$, which in particular implies that μ is p-divisible. As we evidently have $\mu[p^\infty] \simeq G$ by construction, we deduce the desired assertion and complete the proof. □

Definition 2.2.21. Let G be a p-divisible group over R.

(1) We define its *associated formal group law* to be the p-divisible formal group law μ_G over R corresponding to G° under the equivalence in Theorem 2.2.20.
(2) We define its *dimension* to be the dimension of μ_G.

PROPOSITION 2.2.22. Given a p-divisible group G over R of dimension d, its special fiber $\overline{G} := G \times_R k$ is a p-divisible group over k such that $\ker(\varphi_{\overline{G}})$ is finite flat of order p^d.

PROOF. Proposition 1.5.17 implies that $\ker(\varphi_{\overline{G}})$ lies in $\overline{G}^\circ := G^\circ \times_R k$. Hence the assertion follows from Proposition 2.2.16, Lemma 2.2.19, and Theorem 2.2.20. □

THEOREM 2.2.23. Let G be a p-divisible group of height h over R. If write d and $d^\vee$ respectively for the dimensions of G and $G^\vee$, we have $h = d + d^\vee$.

PROOF. Lemma 2.1.5 shows that $\overline{G} := G \times_R k$ is a p-divisible group of order h over k. Let us write $\overline{G} = \varinjlim \overline{G}_v$ where each $\overline{G}_v$ is a finite flat k-group. Proposition 2.1.16 yields the equality $\psi_{\overline{G}} \circ \varphi_{\overline{G}} = [p]_{\overline{G}}$ and in turn implies that $\ker(\varphi_{\overline{G}})$ is a k-subgroup of $\overline{G}[p]$. Moreover, we note by Proposition 2.1.8 that $\varphi_{\overline{G}}$ is surjective. Therefore we obtain a commutative diagram

$$\begin{array}{ccccccccc}
\underline{0} & \longrightarrow & \ker(\varphi_{\overline{G}}) & \longrightarrow & \overline{G} & \xrightarrow{\varphi_{\overline{G}}} & \overline{G}^{(p)} & \longrightarrow & \underline{0} \\
 & & \downarrow & & \downarrow{\scriptstyle [p]_{\overline{G}}} & & \downarrow{\scriptstyle \psi_{\overline{G}}} & & \\
\underline{0} & \longrightarrow & \underline{0} & \longrightarrow & \overline{G} & \xrightarrow{\mathrm{id}} & \overline{G} & \longrightarrow & \underline{0}
\end{array}$$

where the rows are evidently exact. By the snake lemma, the diagram yields an exact sequence

$$\underline{0} \longrightarrow \ker(\varphi_{\overline{G}}) \longrightarrow \overline{G}[p] \longrightarrow \ker(\psi_{\overline{G}}) \longrightarrow \underline{0}.$$

Proposition 2.2.22 shows that $\ker(\varphi_{\overline{G}})$ has order p^d, while Proposition 2.1.8 implies that $\overline{G}[p] \cong G_1$ has order p^h. Hence we deduce from Theorem 1.1.18 that $\ker(\psi_{\overline{G}})$ has order p^{h-d}.

For the desired assertion, it suffices to show that $\ker(\psi_{\overline{G}})$ has order $p^{d^\vee}$. We have

$$\ker(\psi_{\overline{G}}) \cong \ker(\psi_{\overline{G}_1}) \quad \text{and} \quad \ker(\varphi_{\overline{G}^\vee}) \cong \ker(\varphi_{\overline{G}_1^\vee})$$

as easily seen by Proposition 2.1.8 and Proposition 2.1.16. Since the k-groups $\overline{G}_1$ and $\overline{G}_1^{(p)}$ are of the same order by construction, we apply Theorem 1.1.18 with the identifications

$$\psi_{\overline{G}_1}(\overline{G}_1^{(p)}) \cong \overline{G}_1^{(p)} / \ker(\psi_{\overline{G}_1}) \quad \text{and} \quad \operatorname{coker}(\psi_{\overline{G}_1}) \cong \overline{G}_1 / \psi_{\overline{G}_1}(\overline{G}_1^{(p)})$$

to see that $\ker(\psi_{\overline{G}_1})$ and $\operatorname{coker}(\psi_{\overline{G}_1})$ are of the same order. Moreover, Proposition 1.2.13 yields a natural isomorphism $\operatorname{coker}(\psi_{\overline{G}_1}) \cong \ker(\varphi_{\overline{G}_1^\vee})$ as we have $\psi_{\overline{G}_1} = \varphi_{\overline{G}_1^\vee}^\vee$ by definition. Therefore $\ker(\psi_{\overline{G}})$ and $\ker(\varphi_{\overline{G}^\vee})$ have the same order. Now we find $\overline{G}^\vee \cong G^\vee \times_R k$ by Proposition 1.2.5 and in turn deduce from Proposition 2.2.22 that $\ker(\psi_{\overline{G}^\vee})$ has order $p^{d^\vee}$, thereby establishing the desired assertion. $\square$

PROPOSITION 2.2.24. Assume that $R = k$ is an algebraically closed field of characteristic p. Every p-divisible group $G = \varprojlim G_v$ of height 1 over k is isomorphic to either $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ or μ_{p^∞}.

PROOF. Let us first consider the case where G is étale. Each G_v is a finite étale k-group of order p^v with $G_v = G_{v+1}[p^v]$. Since every finite étale k-group is a constant group scheme as noted in Proposition 1.3.7, we find $G_v \simeq \underline{\mathbb{Z}/p^v\mathbb{Z}}$ for each $v \geq 1$ by a simple induction and consequently obtain an isomorphism $G \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p}$.

We now turn to the case where G is not étale. By Proposition 2.1.12, a p-divisible group over R is étale if and only if its dimension is 0. Since the height of G is 1, we deduce from Theorem 2.2.23 that $G^\vee$ is étale and in turn find $G^\vee \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p}$. Hence we obtain an isomorphism $G \simeq \mu_{p^\infty}$ by Proposition 2.1.9 and Example 2.1.11, thereby completing the proof. $\square$

Example 2.2.25. Let E be an ordinary elliptic curve over $\overline{\mathbb{F}}_p$. Since $E[p^\infty]^\circ$ and $E[p^\infty]^{\text{ét}}$ are of height 1 with $E[p]^\circ \simeq \mu_p$ and $E[p]^{\text{ét}} \simeq \underline{\mathbb{Z}/p\mathbb{Z}}$ by Example 1.4.16, there exists an isomorphism

$$E[p^\infty] \simeq \underline{\mathbb{Q}_p/\mathbb{Z}_p} \times \mu_{p^\infty}$$

by Proposition 2.1.12 and Proposition 2.2.24.

2.3. Dieudonné-Manin classification

Throughout this subsection, we assume that $R = k$ is a perfect field of characteristic p. We introduce several algebraic objects and discuss their relation to p-divisible groups over k. We begin by stating the following technical result without a proof.

THEOREM 2.3.1. Let A be a perfect $\mathbb{F}_p$-algebra.

(1) There exists a unique (up to isomorphism) ring $W(A)$ which is p-adically complete with $W(A)/pW(A) \cong A$.
(2) Given a p-adically complete ring B, every homomorphism $\overline{f} : A \to B/pB$ uniquely lifts to a multiplicative map $\hat{f} : A \to B$ and a homomorphism $f : W(A) \to B$.

Remark. For a proof, we refer readers to the book of Serre [**Ser79**, §II.5].

Definition 2.3.2. Let A be a perfect $\mathbb{F}_p$-algebra.

(1) We refer to the ring $W(A)$ in Theorem 2.3.1 as the *ring of Witt vectors over* A.
(2) For each $a \in A$, we define its *Teichmüller lift* $[a] \in W(A)$ to be its image under the unique multiplicative map $A \to W(A)$ which lifts the identity map on A.

Example 2.3.3. We present two important examples which frequently arise in practice.

(1) For an integer $r \geq 1$, the ring $W(\mathbb{F}_{p^r})$ is isomorphic to the valuation ring of the degree r unramified extension of $\mathbb{Q}_p$, as easily seen by Theorem 2.3.1.
(2) The ring $W(\overline{\mathbb{F}}_p)$ is isomorphic to the valuation ring of $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$, where $\widehat{\mathbb{Q}_p^{\mathrm{un}}}$ denotes the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$.

PROPOSITION 2.3.4. Let A be a perfect $\mathbb{F}_p$-algebra.

(1) For every $\alpha \in W(A)$, there exists a unique element $a_0 \in A$ with $\alpha - [a_0] \in pW(A)$.
(2) Every $\alpha \in W(A)$ admits a unique expression $\alpha = \displaystyle\sum_{n=0}^{\infty} [a_n]p^n$ with $a_n \in A$.
(3) The p-th power map on A uniquely lifts to an automorphism $\varphi_{W(A)}$ on $W(A)$ with
$$\varphi_{W(A)}\left(\sum_{n=0}^{\infty} [a_n]p^n\right) = \sum_{n=0}^{\infty} [a_n^p]p^n.$$

PROOF. Statement (1) is evident with a_0 given by the image of α under the natural map $W(A) \twoheadrightarrow W(A)/pW(A) \cong A$. Statement (2) follows from statement (1) by inductively constructing a unique sequence (a_n) in A with
$$\alpha - \sum_{n=0}^{m} [a_n]p^n \in p^m W(A) \quad \text{for each } m \geq 0.$$
Statement (3) is straightforward to verify by Theorem 2.3.1 and the perfectness of A. □

Definition 2.3.5. Let A be a perfect $\mathbb{F}_p$-algebra.

(1) For every $\alpha \in W(A)$, we define its *Teichmüller expansion* to be the unique expression $\alpha = \displaystyle\sum_{n=0}^{\infty} [a_n]p^n$ with $a_n \in A$ given by Proposition 2.3.4.
(2) We call the map $\varphi_{W(A)}$ in Proposition 2.3.4 the *Frobenius automorphism* of $W(A)$.

Remark. Teichmüller expansions for $\mathbb{Z}_p = W(\mathbb{F}_p)$ are not the same as p-adic expansions.

PROPOSITION 2.3.6. Given a perfect $\mathbb{F}_p$-algebra A, an element $\alpha \in W(A)$ is a unit if and only if the first coefficient in the Teichmüller expansion of α is a unit in A.

PROOF. The first coefficient in the Teichmüller expansion of α coincides with the image of α under the natural map $W(A) \twoheadrightarrow W(A)/pW(A) \cong A$. Since $W(A)$ is p-adically complete, the assertion follows from a general fact stated in the Stacks project [**Sta**, Tag 05GI]. $\square$

PROPOSITION 2.3.7. Let A be a perfect $\mathbb{F}_p$-algebra. Take two arbitrary elements $\alpha, \beta \in W(A)$ with Teichmüller expansions $\alpha = \sum_{n=0}^{\infty}[a_n]p^n$ and $\beta = \sum_{n=0}^{\infty}[b_n]p^n$.

(1) The Teichmüller expansion of $\alpha + \beta$ has the first two coefficients given by
$$c_0 = a_0 + b_0 \quad \text{and} \quad c_1 = a_1 + b_1 - W_1\left(a_0^{1/p}, b_0^{1/p}\right),$$
where we write $W_1(t, u) := \dfrac{(t+u)^p - t^p - u^p}{p} \in \mathbb{Z}[t, u]$.

(2) The Teichmüller expansion of $\alpha\beta$ has the first two coefficients given by
$$d_0 = a_0 b_0 \quad \text{and} \quad d_1 = a_0 b_1 + a_1 b_0.$$

PROOF. The addition under the natural surjection $W(A) \twoheadrightarrow W(A)/pW(A) \cong A$ yields the identity $c_0 = a_0 + b_0$. Since every element of A admits a unique p-th root, we have $c_0^{1/p} = a_0^{1/p} + b_0^{1/p}$. Hence we find $[c_0^{1/p}] \in [a_0^{1/p}] + [b_0^{1/p}] + pW(A)$ and in turn get the relation
$$[c_0] = [c_0^{1/p}]^p \in \left([a_0^{1/p}] + [b_0^{1/p}]\right)^p + p^2 W(A).$$
Meanwhile, the addition under the natural map $W(A) \twoheadrightarrow W(A)/p^2W(A)$ yields the relation
$$[c_0] + p[c_1] = [a_0] + [b_0] + p([a_1] + [b_1]) + p^2 W(A).$$
Now we have
$$p[c_1] \in p([a_1] + [b_1]) + [a_0] + [b_0] - \left([a_0^{1/p}] + [b_0^{1/p}]\right)^p + p^2 W(A)$$
and consequently find
$$[c_1] \in [a_1] + [b_1] - W_1\left([a_0^{1/p}], [b_0^{1/p}]\right) + pW(A).$$
We consider the images under the natural surjection $W(A) \twoheadrightarrow W(A)/pW(A) \cong A$ and obtain the identity $c_1 = a_1 + b_1 - W_1\left(a_0^{1/p}, b_0^{1/p}\right)$. Therefore we establish statement (1).

Let us now consider statement (2). The multiplication under the natural surjection $W(A) \twoheadrightarrow W(A)/pW(A) \cong A$ yields the identity $d_0 = a_0 b_0$. Moreover, the multiplication under the natural map $W(A) \twoheadrightarrow W(A)/p^2W(A)$ yields the relation
$$[d_0] + p[d_1] \in [a_0 b_0] + p([a_0 b_1] + [a_1 b_0]) + p^2 W(A).$$
Hence we have
$$p[d_1] \in p([a_0 b_1] + [a_1 b_0]) + p^2 W(A)$$
and consequently find
$$[d_1] \in [a_0 b_1] + [a_1 b_0] + pW(A).$$
We consider the images under the natural surjection $W(A) \twoheadrightarrow W(A)/pW(A) \cong A$ and deduce the identity $d_1 = a_0 b_1 + a_1 b_0$, thereby completing the proof. $\square$

Remark. The book of Serre [**Ser79**, §II.6] explains a way to compute other coefficients in the Teichmüller expansions of $\alpha + \beta$ and $\alpha\beta$.

Our main objective for this subsection is to discuss fundamental theorems of Dieudonné and Manin which describe p-divisible groups over k via certain free $W(k)$-modules. We won't provide their proofs, as these theorems will mostly serve as motivations for some constructions and only play a significant role at the end of Chapter IV. Curious readers may consult the book of Demazure [**Dem72**, Chapters III and IV] for an excellent exposition of these results.

Definition 2.3.8. Let us write σ for the Frobenius automorphism of $W(k)$.

(1) Given $W(k)$-modules D, D' and an integer r, we say that an additive map $f : D \to D'$ is σ^r*-semilinear* if it satisfies the identity

$$f(cm) = \sigma^r(c)f(m) \quad \text{for each } c \in W(k) \text{ and } m \in D.$$

(2) A *Dieudonné module* over $W(k)$ is a finite free $W(k)$-module D with a σ-semilinear endomorphism φ_D, called the *Frobenius endomorphism*, whose image contains pD.

(3) A $W(k)$-linear map $f : D \to D'$ for Dieudonné modules D and D' over $W(k)$ is a *morphism of Dieudonné modules* if it satisfies the identity $f \circ \varphi_D = \varphi_{D'} \circ f$.

Lemma 2.3.9. The ring $W(k)$ is a complete discrete valuation ring with residue field k and uniformizer p.

Proof. Since $W(k)$ is p-adically complete with $W(k)/pW(k) \cong k$ by construction, it is a local ring with maximal ideal $pW(k)$ and residue field k by Proposition 2.3.6 and a general fact stated in the Stacks project [**Sta**, Tag 00E9]. Moreover, Proposition 2.3.4 shows that every element $\alpha \in W(k)$ admits a unique expression $\alpha = p^n u$ with $n \geq 0$ and $u \in W(k)^\times$. Therefore we establish the desired assertion. □

Lemma 2.3.10. Let D be a Dieudonné module over $W(k)$.

(1) The Frobenius endomorphism φ_D is injective.

(2) There exists a unique σ^{-1}-semilinear endomorphism ψ_D on D such that $\varphi_D \circ \psi_D$ and $\psi_D \circ \varphi_D$ coincide with the multiplication by p on D.

Proof. Take $e_1, \cdots, e_r \in D$ which form a basis over $W(k)$. Since $W(k)$ is a principal ideal domain by Lemma 2.3.9, statement (1) follows from the rank-nullity theorem and the fact that $\varphi_D(D)$ has rank r for containing pD. Hence we only need to prove statement (2).

We may write $pe_i = \varphi_D(e_i')$ for a unique element $e_i' \in D$ and in turn obtain a unique σ^{-1}-semilinear endomorphism ψ_D on D with $\varphi_D \circ \psi_D$ being the multiplication by p on D; indeed, ψ_D maps each e_i to e_i'. We wish to prove that $\psi_D \circ \varphi_D$ coincides with the multiplication by p on D. We note that each e_i' satisfies the equality

$$\psi_D(\varphi_D(e_i')) = \psi_D(\varphi_D(\psi_D(e_i))) = \psi_D(pe_i) = pe_i',$$

which means that $\psi_D \circ \varphi_D$ and the multiplication by p agree on the $W(k)$-module $D' \subseteq D$ spanned by $e_1', \cdots, e_r'$. Moreover, D' has rank r as $e_1', \cdots, e_r'$ are linearly independent by construction. Hence we deduce from the rank-nullity theorem that the difference between $\psi_D \circ \varphi_D$ and the multiplication by p vanishes on D, thereby completing the proof. □

Definition 2.3.11. Given a Dieudonné module D over $W(k)$, we refer to the σ^{-1}-semilinear endomorphism ψ_D in Lemma 2.3.10 as the *Verschiebung endomorphism* of D.

Lemma 2.3.12. Given a Dieudonné module D over $W(k)$, its dual $D^\vee = \mathrm{Hom}_{W(k)}(D, W(k))$ is naturally a Dieudonné module over $W(k)$ with

$$\varphi_{D^\vee}(f)(m) = \sigma(f(\psi_D(m))) \quad \text{for all } f \in D^\vee \text{ and } m \in D.$$

Proof. The assertion is straightforward to verify by definition. □

THEOREM 2.3.13 (Dieudonné [**Die55**]). There is an additive anti-equivalence of categories

$$\mathbb{D} : \{\ p\text{-divisible groups over } k\ \} \xrightarrow{\sim} \{\ \text{Dieudonné modules over } W(k)\ \}$$

such that for every p-divisible group G over k we have the following statements:

(1) The rank of $\mathbb{D}(G)$ is equal to the height of G.
(2) The maps φ_G, ψ_G, and $[p]_G$ yield $\varphi_{\mathbb{D}(G)}$, $\psi_{\mathbb{D}(G)}$, and the multiplication by p.
(3) There exists a natural isomorphism $\mathbb{D}(G^\vee) \cong \mathbb{D}(G)^\vee$.

Remark. Let us briefly describe the construction of $\mathbb{D}(G)$ for a p-divisible group $G = \varinjlim G_v$ over k. For each integer $n \geq 1$, we have a k-group W_n with $W_n(A) = W(A)/p^nW(A)$ for every perfect k-algebra A. If $G^\vee$ is connected, $\mathbb{D}(G) := \varprojlim_v \varinjlim_n \mathrm{Hom}_{k\text{-grp}}(G_v, W_n)$ turns out to be a Dieudonné module over $W(k)$ with Frobenius endomorphism induced by φ_G. If $G^\vee$ is étale, it is connected by Theorem 2.2.23 and consequently yields a Dieudonné module $\mathbb{D}(G) := \mathbb{D}(G^\vee)^\vee$ over $W(k)$. In the general case, G admits a natural decomposition

$$G \cong G^{\mathrm{unip}} \times G^{\mathrm{mult}}$$

with $(G^{\mathrm{unip}})^\vee$ connected and $(G^{\mathrm{mult}})^\vee$ étale, thereby giving rise to a Dieudonné module $\mathbb{D}(G) := \mathbb{D}(G^{\mathrm{unip}}) \oplus \mathbb{D}(G^{\mathrm{mult}})$ over $W(k)$.

Definition 2.3.14. We refer to the functor $\mathbb{D}$ in Theorem 2.3.13 as the *Dieudonné functor*.

Example 2.3.15. We describe the Dieudonné functor for some simple p-divisible groups.

(1) $\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})$ is isomorphic to $W(k)$ with $\varphi_{\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})} = \sigma$ and $\psi_{\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p})} = p\sigma^{-1}$.
(2) $\mathbb{D}(\mu_{p^\infty})$ is isomorphic to $W(k)$ with $\varphi_{\mathbb{D}(\mu_{p^\infty})} = p\sigma$ and $\psi_{\mathbb{D}(\mu_{p^\infty})} = \sigma^{-1}$.

Definition 2.3.16. Let us write $K_0(k) := W(k)[1/p]$ for the fraction field of $W(k)$.

(1) We define the *Frobenius automorphism* of $K_0(k)$ to be the unique field automorphism on $K_0(k)$ which extends σ.
(2) An *isocrystal* over $K_0(k)$ is a finite dimensional vector space D over $K_0(k)$ with a σ-semilinear automorphism φ_D called the *Frobenius automorphism* of D.
(3) A $K_0(k)$-linear map $g : D \to D'$ for isocrystals D and D' over $K_0(k)$ is a *morphism of isocrystals* if it satisfies the identity

$$g(\varphi_D(v)) = \varphi_{D'}(g(v)) \quad \text{for each } v \in D.$$

LEMMA 2.3.17. Let σ denote the Frobenius automorphism of $K_0(k)$.

(1) Every Dieudonné module D over $W(k)$ yields an isocrystal $D[1/p] = D \otimes_{W(k)} K_0(k)$ over $K_0(k)$ with Frobenius automorphism $\varphi_D \otimes 1$.
(2) Given an isocrystal D over $K_0(k)$, its dual $D^\vee = \mathrm{Hom}_{K_0(k)}(D, K_0(k))$ is naturally an isocrystal over $K_0(k)$ with

$$\varphi_{D^\vee}(f)(v) = \sigma(f(\varphi_D^{-1}(v))) \quad \text{for all } f \in D^\vee \text{ and } v \in D.$$

(3) Given two isocrystals D and D' over $K_0(k)$, their tensor product $D \otimes_{K_0(k)} D'$ is naturally an isocrystal over $K_0(k)$ with Frobenius automorphism $\varphi_D \otimes \varphi_{D'}$.

PROOF. All statements are straightforward to verify by definition. □

Example 2.3.18. For an isocrystal D of rank r over $K_0(k)$, its determinant $\det(D) = \wedge^r(D)$ is naturally an isocrystal of rank 1 over $K_0(k)$ as easily seen by Lemma 2.3.17.

Definition 2.3.19. We say that a homomorphism of group schemes or p-divisible groups is an *isogeny* if it is surjective with finite flat kernel.

Example 2.3.20. We present some examples of isogenies between p-divisible groups.

(1) Given a p-divisible group G over k, the maps $[p]_G, \varphi_G$, and ψ_G are all isogenies by Proposition 2.1.8 and Proposition 2.1.16.
(2) An isogeny $A \to B$ of two abelian varieties over k induces an isogeny $A[p^\infty] \to B[p^\infty]$.

PROPOSITION 2.3.21. A homomorphism $f : G \to H$ of p-divisible groups over k is an isogeny if and only if it induces an isomorphism $\mathbb{D}(H)[1/p] \simeq \mathbb{D}(G)[1/p]$.

PROOF. Let us first assume that f is an isogeny. Its kernel lies in G_v for some $v \geq 1$ and thus is a p-power torsion k-group. Theorem 2.3.13 implies that the map $\mathbb{D}(H) \to \mathbb{D}(G)$ induced by f is injective with its cokernel killed by a power of p. We deduce that f induces an isomorphism $\mathbb{D}(H)[1/p] \simeq \mathbb{D}(G)[1/p]$.

For the converse, we now assume that f induces an isomorphism $\mathbb{D}(H)[1/p] \simeq \mathbb{D}(G)[1/p]$. The map $\mathbb{D}(H) \to \mathbb{D}(G)$ is injective with $\mathbb{D}(H)$ and $\mathbb{D}(G)$ having the same rank over $W(k)$. The cokernel of this map is a p-power torsion $W(k)$-module by Lemma 2.3.9. Hence we deduce from Theorem 2.3.13 that f is an isogeny as desired. □

Definition 2.3.22. Let D be a nonzero isocrystal over $K_0(k)$.

(1) The *degree* of D is the unique integer $\deg(D)$ with $\varphi_{\det(D)}(1) \in p^{\deg(D)} W(k)^\times$, where we fix an isomorphism $\det(D) \simeq W(k)$.
(2) We write $\mathrm{rk}(D)$ for the rank of D and define the *slope* of D to be $\mu(D) := \dfrac{\deg(D)}{\mathrm{rk}(D)}$.

Example 2.3.23. Let $\lambda = d/r$ be a rational number written in lowest terms with $r > 0$. The *simple isocrystal of slope* λ over $K_0(k)$ is an isocrystal D_λ over $K_0(k)$ of rank r with

$$\varphi_{D_\lambda}(e_1) = e_2, \cdots, \varphi_{D_\lambda}(e_{r-1}) = e_r, \varphi_{D_\lambda}(e_r) = p^d e_1,$$

where $e_1, \cdots, e_r$ are basis vectors. It is evident that D_λ has rank r, degree d, and slope λ.

PROPOSITION 2.3.24. Given a p-divisible group G over k of height h and dimension d, the associated isocrystal $\mathbb{D}(G)[1/p]$ over $K_0(k)$ has rank h and degree d.

PROOF. As noted in Proposition 2.2.22 and Example 2.3.20, the Frobenius φ_G is an isogeny with $\ker(\varphi_G)$ having order p^d. Moreover, Proposition 2.1.16 implies that $\ker(\varphi_G)$ is p-torsion. Hence we deduce from Lemma 2.3.9 and Theorem 2.3.13 that $\varphi_{\mathbb{D}(G)}$ is injective with $\mathrm{coker}(\varphi_{\mathbb{D}(G)}) \simeq (W(k)/pW(k))^{\oplus d}$. Now it is straightforward to verify that $\mathbb{D}(G)[1/p]$ has degree d. Since $\mathbb{D}(G)[1/p]$ evidently has rank h over $K_0(k)$ by Theorem 2.3.13, we establish the desired assertion. □

THEOREM 2.3.25 (Manin [**Man63**]). Every isocrystal D over $K_0(\overline{k})$ admits a direct sum decomposition

$$D \simeq \bigoplus_{i=1}^{n} D_{\lambda_i}^{\oplus m_i} \quad \text{with } \lambda_i \in \mathbb{Q}.$$

Example 2.3.26. If an elliptic curve E over $\overline{\mathbb{F}}_p$ is ordinary, we have

$$\mathbb{D}(E[p^\infty])[1/p] \simeq D_0 \oplus D_1$$

as easily seen by Example 2.2.25 and Example 2.3.15.

Remark. If E is supersingular, $\mathbb{D}(E[p^\infty])[1/p]$ turns out to be isomorphic to $D_{1/2}$.

3. Hodge-Tate decomposition

In this section, we finally enter the realm of p-adic Hodge theory. Assuming some technical results, we prove the Hodge-Tate decomposition for Tate modules of p-divisible groups. The primary reference for this section is the article of Tate [**Tat67**].

3.1. Tate twists of p-adic representations

In this subsection, we introduce some basic notions in p-adic Hodge theory, such as p-adic fields, p-adic representations and their Tate twists. Given a valued field L, we write $\mathcal{O}_L$ for its valuation ring, $\mathfrak{m}_L$ for its maximal ideal, and k_L for its residue field.

Definition 3.1.1. Let E be an arbitrary field.

(1) A *p-adic Γ_E-representation* is a finite dimensional $\mathbb{Q}_p$-vector space V together with a continuous homomorphism $\Gamma_E \to \mathrm{GL}(V)$.
(2) The *p-adic cyclotomic character* of E is the character $\chi_E : \Gamma_E \to \mathbb{Z}_p^\times$ via which Γ_E acts on $\mathbb{Z}_p(1) := T_p(\mu_{p^\infty}) = \varprojlim \mu_{p^v}(\overline{E})$.
(3) Given an integer n, the *n-fold Tate twist* of a $\mathbb{Z}_p[\Gamma_E]$-module M is

$$M(n) := \begin{cases} M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1)^{\otimes n} & \text{for } n \geq 0, \\ M \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p(1)^\vee)^{\otimes -n} & \text{for } n < 0. \end{cases}$$

Example 3.1.2. Let E be an arbitrary field.

(1) Given a p-divisible group G over E, its *rational Tate module* $V_p(G) := T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a p-adic Γ_E-representation by Proposition 2.1.18.
(2) For a proper smooth variety X over E, the étale cohomology group $H^n_{\text{ét}}(X_{\overline{E}}, \mathbb{Q}_p)$ is a p-adic Γ_E-representation.

LEMMA 3.1.3. Let E be an arbitrary field and M be a $\mathbb{Z}_p[\Gamma_E]$-module.

(1) There exist natural Γ_E-equivariant isomorphisms

$$M(n) \cong M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(n) \quad \text{and} \quad M(n)^\vee \cong M^\vee(-n) \quad \text{for each } n \in \mathbb{Z}.$$

(2) If Γ_E acts on M via a homomorphism $\rho : \Gamma_E \to \mathrm{Aut}(M)$, it acts on $M(n)$ for each $n \in \mathbb{Z}$ via $\chi_E^n \cdot \rho$.

PROOF. Statement (1) is straightforward to verify by definition. Statement (2) is an immediate consequence of statement (1). □

Definition 3.1.4. A *p-adic field* is an extension of $\mathbb{Q}_p$ which is discretely valued and complete with a perfect residue field of characteristic p.

Example 3.1.5. We present some essential examples of p-adic fields.

(1) Every finite extension of $\mathbb{Q}_p$ is a p-adic field.
(2) Every perfect field k of characteristic p gives rises to a p-adic field $K_0(k) = W(k)[1/p]$ as noted in Lemma 2.3.9.

Remark. We will see in Chapter III, Proposition 2.2.19 that every p-adic field is a finite extension of $K_0(k)$ for some perfect field k of characteristic p. There are p-adic fields which are not algebraic over $\mathbb{Q}_p$, such as $\widehat{\mathbb{Q}_p^{\text{un}}} := K_0(\overline{\mathbb{F}}_p)$. It is worthwhile to mention that for many authors p-adic fields simply mean finite extensions of $\mathbb{Q}_p$.

For the rest of this section, we let K be a p-adic field. In addition, we write $\mathfrak{m}$ for its maximal ideal, k for its residue field, and χ for its p-adic cyclotomic character.

Definition 3.1.6. The *completed algebraic closure* of K, denoted by $\mathbb{C}_K$, is the p-adic completion of $\overline{K}$.

Remark. The field $\mathbb{C}_K$ is not a p-adic field as its valuation is not discrete.

Example 3.1.7. If K is a finite extension of $\mathbb{Q}_p$, we often write $\mathbb{C}_p = \mathbb{C}_K$ and refer to it as the *field of p-adic complex numbers*.

LEMMA 3.1.8. The action of Γ_K on $\overline{K}$ uniquely extends to a continuous action on $\mathbb{C}_K$.

PROOF. The assertion is evident by the continuity of the Γ_K-action on $\overline{K}$. □

Definition 3.1.9. The *normalized p-adic valuation* on $\mathbb{C}_K$ is the unique valuation ν on $\mathbb{C}_K$ with $\nu(p) = 1$.

PROPOSITION 3.1.10. The field $\mathbb{C}_K$ is algebraically closed.

PROOF. We wish to prove that every nonconstant polynomial $f(t)$ over $\mathbb{C}_K$ admits a root in $\mathbb{C}_K$. We may replace $f(t)$ by $p^{md}f(t/p^m)$ for some sufficiently large $m \in \mathbb{Z}$ to assume that $f(t)$ is a polynomial over $\mathcal{O}_{\mathbb{C}_K}$. Let us write

$$f(t) = t^d + c_1t^{d-1} + \cdots + c_d \quad \text{with } c_i \in \mathcal{O}_{\mathbb{C}_K}.$$

For each integer $n \geq 1$, we choose a polynomial

$$f_n(t) = t^d + c_{1,n}t^{d-1} + \cdots + c_{d,n}$$

with $c_{i,n} \in \mathcal{O}_{\overline{K}}$ and $\nu(c_i - c_{i,n}) \geq dn$. Since $\mathcal{O}_{\overline{K}}$ is integrally closed, each $f_n(t)$ admits a factorization into linear polynomials over $\mathcal{O}_{\overline{K}}$; in other words, we have

$$f_n(t) = \prod_{i=1}^{d}(t - \beta_{n,i}) \quad \text{with } \beta_{n,i} \in \mathcal{O}_{\overline{K}}. \tag{3.1}$$

Let us construct a sequence (α_n) in $\mathcal{O}_{\overline{K}}$ with $f_n(\alpha_n) = 0$ and $\nu(\alpha_n - \alpha_{n-1}) \geq n-1$. We set $\alpha_1 := \beta_{1,1} \in \mathcal{O}_{\overline{K}}$ and proceed by induction on n. We have

$$f_n(\alpha_{n-1}) = f_n(\alpha_{n-1}) - f_{n-1}(\alpha_{n-1}) = \sum_{i=1}^{d}(c_{i,n} - c_{i,n-1})\alpha_{n-1}^{d-i}$$

and in turn find $\nu(f_n(\alpha_{n-1})) \geq d(n-1)$ as each $c_{i,n} - c_{i,n-1} = (c_{i,n} - c_i) + (c_i - c_{i,n-1})$ has valuation at least $d(n-1)$. We deduce from the identity (3.1) that $f_n(t)$ admits a root $\alpha_n = \beta_{n,i} \in \mathcal{O}_{\overline{K}}$ with $\nu(\alpha_{n-1} - \alpha_n) \geq n-1$.

The sequence (α_n) is Cauchy by construction and thus converges to an element $\alpha \in \mathcal{O}_{\mathbb{C}_K}$. Moreover, for each integer $n \geq 1$ we obtain the identity

$$f(\alpha_n) = f(\alpha_n) - f_n(\alpha_n) = \sum_{i=1}^{d}(c_i - c_{i,n})\alpha_n^{d-i}$$

and in turn find $\nu(f(\alpha_n)) \geq dn$. Hence we see that α is a root of $f(t)$, thereby completing the proof. □

Remark. We can alternatively derive Proposition 3.1.10 from Krasner's lemma by modifying our argument. Moreover, we can use Krasner's lemma to show that $\overline{K}$ is not complete; in particular, we have $\mathbb{C}_K \neq \overline{K}$.

We assume the following fundamental result about the Tate twists of $\mathbb{C}_K$.

THEOREM 3.1.11 (Tate [**Tat67**], Sen [**Sen80**]). For the Galois cohomology of $\mathbb{C}_K$ and its Tate twists, we have the following statements:

(1) $H^0(\Gamma_K, \mathbb{C}_K)$ admits a natural isomorphism $H^0(\Gamma_K, \mathbb{C}_K) \cong K$.
(2) $H^1(\Gamma_K, \mathbb{C}_K)$ is an 1-dimensional vector space over K.
(3) $H^0(\Gamma_K, \mathbb{C}_K(n))$ and $H^1(\Gamma_K, \mathbb{C}_K(n))$ vanish for $n \neq 0$.

Remark. We refer curious readers to the notes of Brinon-Conrad [**BC**, §14] for a proof, which involves the higher ramification theory and the local class field theory.

PROPOSITION 3.1.12. Every p-adic Γ_K-representation V yields a natural $\mathbb{C}_K$-linear map

$$\tilde{\alpha}_V : \bigoplus_{n\in\mathbb{Z}} \left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K \mathbb{C}_K(n) \to V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$$

which is Γ_K-equivariant and injective.

PROOF. For each $n \in \mathbb{Z}$, we have a Γ_K-equivariant injective K-linear map

$$\tilde{\alpha}_{V,K}^{(n)} : \left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K K(n) \hookrightarrow V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n) \otimes_K K(n) \cong V \otimes_{\mathbb{Q}_p} \mathbb{C}_K.$$

Let us extend each $\tilde{\alpha}_{V,K}^{(n)}$ to a Γ_K-equivariant $\mathbb{C}_K$-linear map

$$\tilde{\alpha}_V^{(n)} : \left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K \mathbb{C}_K(n) \to V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$$

and set $\tilde{\alpha}_V := \bigoplus_{n\in\mathbb{Z}} \tilde{\alpha}_V^{(n)}$. We wish to show that $\tilde{\alpha}_V$ is injective.

Suppose for contradiction that $\ker(\tilde{\alpha}_V)$ is nonzero. For every $n \in \mathbb{Z}$, we take a K-basis $(v_{m,n})$ of $\left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \otimes_K K(n)$ and regard each $v_{m,n}$ as a vector in $V \otimes_{\mathbb{Q}_p} \mathbb{C}_K$ via the map $\tilde{\alpha}_{V,K}^{(n)}$. In addition, we choose a nontrivial $\mathbb{C}_K$-linear relation $\sum c_{m,n} v_{m,n} = 0$ with minimum number of nonzero terms. We may set $c_{m_0,n_0} = 1$ for some integers m_0 and n_0. For every $\gamma \in \Gamma_K$, we apply Lemma 3.1.3 to find

$$0 = \gamma\left(\sum c_{m,n} v_{m,n}\right) - \chi(\gamma)^{n_0}\left(\sum c_{m,n} v_{m,n}\right) = \sum \left(\gamma(c_{m,n})\chi(\gamma)^n - \chi(\gamma)^{n_0} c_{m,n}\right) v_{m,n}.$$

Since the coefficient of v_{m_0,n_0} in the last expression is 0, the minimality of our linear relation implies that all coefficients in the last expression must vanish and in turn yields the relation

$$\gamma(c_{m,n})\chi(\gamma)^{n-n_0} = c_{m,n} \quad \text{for every } \gamma \in \Gamma_K.$$

Now Lemma 3.1.3 and Theorem 3.1.11 together imply that each $c_{m,n}$ lies in K with $c_{m,n} = 0$ for $n \neq n_0$. Hence we have a nontrivial K-linear relation $\sum c_{m,n_0} v_{m,n_0} = 0$ on the basis (v_{m,n_0}) of $\left(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n_0)\right)^{\Gamma_K} \otimes_K K(n_0)$, thereby obtaining a desired contradiction. □

Definition 3.1.13. We say that a p-adic Γ_K-representation V is *Hodge-Tate* if the natural map $\tilde{\alpha}_V$ in Proposition 3.1.12 is an isomorphism.

Example 3.1.14. Every Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ is Hodge-Tate by Theorem 3.1.11.

Remark. We will see in §3.3 that the rational Tate-module of a p-divisible group over $\mathcal{O}_K$ is always Hodge-Tate.

3.2. Points on p-divisible groups

For the rest of this section, we take the base ring to be $R = \mathcal{O}_K$. The main objective for this subsection is to investigate points on p-divisible groups over $\mathcal{O}_K$. We let L denote the p-adic completion of an algebraic extension of K. A primary example of such a field is $\mathbb{C}_K$.

LEMMA 3.2.1. The valuation ring $\mathcal{O}_L$ is $\mathfrak{m}$-adically complete; in other words, there exists a natural isomorphism

$$\mathcal{O}_L \cong \varprojlim \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L.$$

PROOF. The ideal $\mathfrak{m}$ contains p as the residue field $k = \mathcal{O}_K/\mathfrak{m}$ is of characteristic p. Since $\mathcal{O}_K$ is a discrete valuation ring, we deduce that the p-adic topology coincides with the $\mathfrak{m}$-adic topology and consequently establish the desired assertion by observing that $\mathcal{O}_L$ is p-adically complete. □

Definition 3.2.2. Given a p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$, we define its *group of $\mathcal{O}_L$-valued points* to be

$$G(\mathcal{O}_L) := \varprojlim_i \varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

Remark. Readers should be aware that $G(\mathcal{O}_L)$ is in general not equal to $\varinjlim_v G_v(\mathcal{O}_L)$. This subtlety comes from the fact that we take points on G as a formal $\mathcal{O}_K$-group. In fact, if we write $G_v = \mathrm{Spec}\,(A_v)$ for each $v \geq 1$, we argue as in Lemma 2.2.19 to naturally identify G with a formal $\mathcal{O}_K$-group $\mathscr{G} = \mathrm{Spf}(\varprojlim A_v)$ and find $G(\mathcal{O}_L) \cong \mathscr{G}(\mathcal{O}_L)$.

Example 3.2.3. We describe the $\mathcal{O}_L$-valued points for some p-divisible groups of height 1.

(1) The p-power roots of unity μ_{p^∞} admits a natural isomorphism

$$\mu_{p^\infty}(\mathcal{O}_L) \cong 1 + \mathfrak{m}_L.$$

In fact, since $\mathfrak{m}_L$ contains p, we identify $\varinjlim_v \mu_{p^v}(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$ with the image of $1+\mathfrak{m}_L$ in $\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L$ and thus obtain the desired isomorphism by Lemma 3.2.1.

(2) The constant p-divisible group $\underline{\mathbb{Q}_p/\mathbb{Z}_p}$ admits a natural isomorphism

$$\underline{\mathbb{Q}_p/\mathbb{Z}_p}(\mathcal{O}_L) \cong \mathbb{Q}_p/\mathbb{Z}_p.$$

In fact, since $\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L$ is connected, we have $\underline{\mathbb{Z}/p^v\mathbb{Z}}(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \cong \mathbb{Z}/p^v\mathbb{Z}$ and thus obtain the desired isomorphism.

PROPOSITION 3.2.4. Given a p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$, the group $G(\mathcal{O}_L)$ is naturally a $\mathbb{Z}_p$-module such that its torsion part $G(\mathcal{O}_L)_{\mathrm{tors}}$ admits a natural identification

$$G(\mathcal{O}_L)_{\mathrm{tors}} \cong \varinjlim_v \varprojlim_i G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L).$$

PROOF. Proposition 2.1.8 shows that each $\varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$ is a $\mathbb{Z}_p$-module and in turn implies that $G(\mathcal{O}_L)$ is also a $\mathbb{Z}_p$-module. Therefore $G(\mathcal{O}_L)_{\mathrm{tors}}$ consists of p-power torsions. In addition, we observe by Proposition 2.1.8 that the p^v-torsion part of each $\varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$ is $G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$. Since filtered colimits are exact in the category of abelian groups as stated in the Stacks project [**Sta**, Tag 04B0], we deduce that the p^v-torsion part of $G(\mathcal{O}_L)$ is $\varprojlim_i G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$. The desired assertion is now evident. □

Proposition 3.2.5. Given a p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$ with $G_v = \operatorname{Spec}(A_v)$, there exists a canonical isomorphism

$$G(\mathcal{O}_L) \cong \operatorname{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\varprojlim A_v, \mathcal{O}_L).$$

Proof. For every continuous $\mathcal{O}_K$-algebra homomorphism $f : \varprojlim A_v \to \mathcal{O}_L$, the induced map $f_i : \varprojlim A_v \to \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L$ for each $i \geq 1$ factors through a natural surjection $\varprojlim A_v \twoheadrightarrow A_{w_i}$ for some $w_i \geq 1$. Hence we have a canonical map

$$\operatorname{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\varprojlim A_v, \mathcal{O}_L) \longrightarrow \varprojlim_i \varinjlim_v \operatorname{Hom}_{\mathcal{O}_K}(A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$$

which sends each $f \in \operatorname{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\varprojlim A_v, \mathcal{O}_L)$ to $(f_i) \in \varprojlim_i \varinjlim_v \operatorname{Hom}_{\mathcal{O}_K}(A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L)$. It is not hard to see that this map is an isomorphism by Lemma 3.2.1. Now we obtain the desired isomorphism from the natural identification

$$G(\mathcal{O}_L) \cong \varprojlim_i \varinjlim_v \operatorname{Hom}_{\mathcal{O}_K}(A_v, \mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L),$$

thereby completing the proof. □

Remark. Proposition 3.2.5 establishes a canonical isomorphism $G(\mathcal{O}_L) \cong \mathscr{G}(\mathcal{O}_L)$ for the formal $\mathcal{O}_K$-group $\mathscr{G} = \operatorname{Spf}(\varprojlim A_v)$.

Proposition 3.2.6. Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$.

(1) If G is connected of dimension d, it admits a $\mathbb{Z}_p$-module isomorphism

$$G(\mathcal{O}_L) \simeq \operatorname{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathcal{O}_K[[t_1, \cdots, t_d]], \mathcal{O}_L)$$

where the multiplication by p on the target is induced by $[p]_{\mu_G}$.

(2) If G is étale, $G(\mathcal{O}_L)$ is torsion with a natural isomorphism $G(\mathcal{O}_L) \cong \varinjlim G_v(\mathcal{O}_L/\mathfrak{m}\mathcal{O}_L)$.

Proof. Statement (1) is evident by Lemma 2.2.19 and Proposition 3.2.5. Let us now assume for statement (2) that G is étale. Each G_v is formally étale by a general fact stated in the Stacks project [**Sta**, Tag 02HM]; in particular, there exists a natural isomorphism $G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \cong G_v(\mathcal{O}_L/\mathfrak{m}^{i+1}\mathcal{O}_L)$ for each integer $i \geq 1$. Hence we find

$$G(\mathcal{O}_L) = \varprojlim_i \varinjlim_v G_v(\mathcal{O}_L/\mathfrak{m}^i\mathcal{O}_L) \cong \varinjlim G_v(\mathcal{O}_L/\mathfrak{m}\mathcal{O}_L)$$

and in turn deduce from Proposition 2.1.8 that $G(\mathcal{O}_L)$ is a torsion group. □

Remark. If L is a finite extension of K, we have $\mathfrak{m}\mathcal{O}_L = \mathfrak{m}_L^j$ for some integer $j \geq 1$ and thus find $G^{\text{ét}}(\mathcal{O}_L) \cong \varinjlim G_v^{\text{ét}}(\mathcal{O}_L/\mathfrak{m}\mathcal{O}_L) \cong \varinjlim G_v^{\text{ét}}(\mathcal{O}_L/\mathfrak{m}_L) \cong \varinjlim G_v^{\text{ét}}(k_L)$ where the second isomorphism follows from the fact that each $G_v^{\text{ét}}$ is formally étale as noted in the proof.

Lemma 3.2.7. An $\mathcal{O}_K$-algebra homomorphism $f : \mathcal{O}_K[[t_1, \cdots, t_n]] \to L$ is continuous if and only if each $f(t_i)$ lies in $\mathfrak{m}_L$.

Proof. The map f is continuous if and only if there exists an integer v with $f(t_i^v) \in \mathfrak{m}_L$ for each $i = 1, \cdots, n$. Hence the assertion follows from the fact that $\mathcal{O}_K$ is reduced. □

Remark. Proposition 3.2.6 and Lemma 3.2.7 together show that every p-divisible group G over $\mathcal{O}_K$ of dimension d gives rise to an isomorphism $G^\circ(\mathcal{O}_L) \simeq \mathfrak{m}_L^{\oplus d}$ with group law on $\mathfrak{m}_L^d$ induced by μ_G. It turns out that the multiplication and the inverse on $\mathfrak{m}_L^{\oplus d}$ are analytic functions; in other words, $G^\circ(\mathcal{O}_L) \simeq \mathfrak{m}_L^{\oplus d}$ is a p-adic analytic group.

PROPOSITION 3.2.8. Every p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$ yields an exact sequence

$$0 \longrightarrow G^\circ(\mathcal{O}_L) \longrightarrow G(\mathcal{O}_L) \longrightarrow G^{\text{ét}}(\mathcal{O}_L) \longrightarrow 0.$$

PROOF. The sequence is left exact as limits and filtered colimits are left exact in the category of abelian groups. Hence we only need show that the map $G(\mathcal{O}_L) \to G^{\text{ét}}(\mathcal{O}_L)$ is surjective. For each integer $v \geq 1$, we let A_v, A_v°, and $A_v^{\text{ét}}$ respectively denote the affine rings of G_v, G_v°, and $G_v^{\text{ét}}$. In addition, we write $\mathscr{A} := \varprojlim A_v$, $\mathscr{A}^\circ := \varprojlim A_v^\circ$, and $\mathscr{A}^{\text{ét}} := \varprojlim A_v^{\text{ét}}$. By Proposition 3.2.5, it suffices to prove the surjectivity of the map

$$\text{Hom}_{\mathcal{O}_K-\text{cont}}(\mathscr{A}, \mathcal{O}_L) \to \text{Hom}_{\mathcal{O}_K-\text{cont}}(\mathscr{A}^{\text{ét}}, \mathcal{O}_L). \tag{3.2}$$

Lemma 2.2.19 yields a topological $\mathcal{O}_K$-algebra isomorphism

$$\mathscr{A}^\circ \simeq \mathcal{O}_K[[t_1, \cdots, t_d]]$$

where d denotes the dimension of G. Since k is perfect, we apply Proposition 1.4.15 to obtain a topological k-algebra isomorphism

$$(\mathscr{A}^{\text{ét}} \otimes_{\mathcal{O}_K} k)[[t_1, \cdots, t_d]] \simeq (\mathscr{A}^\circ \otimes_{\mathcal{O}_K} k) \widehat{\otimes}_k (\mathscr{A}^{\text{ét}} \otimes_{\mathcal{O}_K} k) \cong \mathscr{A} \otimes_{\mathcal{O}_K} k.$$

By Lemma 2.2.18, this map lifts to a surjective $\mathcal{O}_K$-algebra homomorphism

$$\theta : \mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]] \longrightarrow \mathscr{A}.$$

Moreover, Lemma 2.2.18 shows that $\mathscr{A}$ is flat over $\mathcal{O}_K$ and consequently yields the relation $\ker(\theta) \otimes_{\mathcal{O}_K} k = 0$ by a general fact stated in the Stacks project [**Sta**, Tag 00HL]. For each $v \geq 1$, we take an ideal $\mathscr{J}_v$ of $\mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]]$ with $\mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]]/\mathscr{J}_v \cong A_v^\circ \otimes_{\mathcal{O}_K} A_v^{\text{ét}}$ and obtain a short exact sequence

$$0 \longrightarrow \ker(\theta)/\ker(\theta) \cap \mathscr{J}_v \longrightarrow \mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]]/\mathscr{J}_v \longrightarrow \mathscr{A}/\theta(\mathscr{J}_v) \longrightarrow 0.$$

We have $\mathfrak{m}\,(\ker(\theta)/\ker(\theta) \cap \mathscr{J}_v) = \ker(\theta)/\ker(\theta) \cap \mathscr{J}_v$ and thus find $\ker(\theta) = \ker(\theta) \cap \mathscr{J}_v$ for each $v \geq 1$ by Lemma 2.2.17 as $\mathscr{A}^{\text{ét}}[[t_1, \cdots, t_d]]/\mathscr{J}_v \cong A_v^\circ \otimes_{\mathcal{O}_K} A_v^{\text{ét}}$ is noetherian. Since we have $\bigcap \mathscr{J}_v = 0$, we see that $\ker(\theta)$ is trivial and in turn deduce that θ is an isomorphism.

The map θ is continuous as the kernel of each $\theta_v : \mathscr{A} \to A_v$ is open by the fact that the R-algebra A_v is of finite length. Moreover, with θ being a topological isomorphism after base change to k, we observe that every power of the ideal $\mathscr{I} := (t_1, \cdots, t_d)$ contains an open set in its image under θ and in turn find that θ is open. Hence θ is a topological R-algebra isomorphism. Now θ yields a surjective continuous map $\mathscr{A} \twoheadrightarrow \mathscr{A}^{\text{ét}}$ which splits the natural map $\mathscr{A}^{\text{ét}} \to \mathscr{A}$. We conclude that the map (3.2) is surjective as desired. □

PROPOSITION 3.2.9. Let G be a p-divisible group over $\mathcal{O}_K$.

(1) For every $g \in G(\mathcal{O}_L)$, we have $p^n g \in G^\circ(\mathcal{O}_L)$ for each $n \gg 0$.
(2) If L is algebraically closed, $G(\mathcal{O}_L)$ is p-divisible in the sense that the multiplication by p on $G(\mathcal{O}_L)$ is surjective.

PROOF. Since statement (1) is an immediate consequence of Proposition 3.2.6 and Proposition 3.2.8, we only need to establish statement (2). In light of Proposition 3.2.8, it suffices to show that the multiplication by p is surjective on each $G^{\text{ét}}(\mathcal{O}_L)$ and $G^\circ(\mathcal{O}_L)$. The surjectivity on $G^{\text{ét}}(\mathcal{O}_L)$ follows from Proposition 2.1.8 and Proposition 3.2.6. Moreover, we deduce the surjectivity on $G^\circ(\mathcal{O}_L)$ from Proposition 3.2.6 and the p-divisibility of μ_G. □

Remark. If L is not algebraically closed, for every $g \in G(\mathcal{O}_L)$ we have a finite extension L' of L and an element $h \in G(\mathcal{O}_{L'})$ with the equality $g = ph$, where we naturally identify $G(\mathcal{O}_L)$ as a subgroup of $G(\mathcal{O}_{L'})$.

Definition 3.2.10. Let G be a p-divisible group over $\mathcal{O}_K$ and M be an $\mathcal{O}_K$-module. We write $\mathscr{I}$ for the augmentation ideal of μ_G.

(1) The *tangent space of* G *with values in* M is $t_G(M) := \mathrm{Hom}_{\mathcal{O}_K\text{-mod}}(\mathscr{I}/\mathscr{I}^2, M)$.
(2) The *cotangent space of* G *with values in* M is $t_G^*(M) := \mathscr{I}/\mathscr{I}^2 \otimes_{\mathcal{O}_K} M$.

Remark. We may naturally identify t_G and t_G^* respectively with the tangent space and the cotangent space of the formal group $\mathscr{G}\mu_G$ associated to μ_G.

PROPOSITION 3.2.11. For a p-divisible group G over $\mathcal{O}_K$ of dimension d, both $t_G(L)$ and $t_G^*(L)$ are vector spaces over L of dimension d.

PROOF. We identify the augmentation ideal of μ_G with $\mathscr{I} := (t_1, \cdots, t_d) \subseteq \mathcal{O}_K[[t_1, \cdots, t_d]]$ and obtain the assertion by observing that $\mathscr{I}/\mathscr{I}^2$ is a free $\mathcal{O}_K$-module of rank d. □

Definition 3.2.12. Given a p-divisible group G over $\mathcal{O}_K$ of dimension d, we define the *valuation filtration* on the group $G^\circ(\mathcal{O}_L)$ to be the collection $\left\{ \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) \right\}_{\lambda>0}$ with

$$\mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) := \{ f \in G^\circ(\mathcal{O}_L) : \nu(f(\alpha)) \geq \lambda \text{ for each } \alpha \in \mathscr{I} \},$$

where we write $\mathscr{I}$ for the augmentation ideal of μ_G and fix a $\mathbb{Z}_p$-module isomorphism $G^\circ(\mathcal{O}_L) \simeq \mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathcal{O}_K[[t_1, \cdots, t_d]], \mathcal{O}_L)$ given by Proposition 3.2.6.

Remark. It is not hard to see that the collection $\left\{ \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) \right\}_{\lambda>0}$ does not depend on the choice of the isomorphism $G^\circ(\mathcal{O}_L) \simeq \mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathcal{O}_K[[t_1, \cdots, t_d]], \mathcal{O}_L)$. If we take an isomorphism $G^\circ(\mathcal{O}_L) \simeq \mathfrak{m}_L^{\oplus d}$ as remarked after Lemma 3.2.7, for each $\lambda > 0$ we have $\mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) \simeq \mathfrak{m}_{L,\lambda}^{\oplus d}$ with $\mathfrak{m}_{L,\lambda} := \{ c \in \mathcal{O}_L : \nu(c) \geq \lambda \}$.

LEMMA 3.2.13. Given a p-divisible group G over $\mathcal{O}_K$, we have

$$\bigcup_{\lambda>0} \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) = G^\circ(\mathcal{O}_L) \quad \text{and} \quad \bigcap_{\lambda>0} \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) = 0.$$

PROOF. The assertion is evident by Lemma 3.2.7 and the completeness of $\mathcal{O}_L$. □

LEMMA 3.2.14. Let G be a p-divisible group over $\mathcal{O}_K$ and λ be a positive real number. For every $f \in \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L)$, we have $pf \in \mathrm{Fil}^\kappa G^\circ(\mathcal{O}_L)$ with $\kappa = \min(\lambda + 1, 2\lambda)$.

PROOF. Let $\mathscr{I}$ denote the augmentation ideal of μ_G and take an arbitrary element $\alpha \in \mathscr{I}$. We may write $[p]_{\mu_G}(\alpha) = p\alpha + \beta$ for some $\beta \in \mathscr{I}^2$ by Lemma 2.2.13 and in turn find

$$(pf)(\alpha) = f([p]_{\mu_G}(\alpha)) = f(p\alpha + \beta) = pf(\alpha) + f(\beta).$$

Therefore we have $\nu((pf)(\alpha)) \geq \min(\lambda + 1, 2\lambda)$ as desired. □

LEMMA 3.2.15. Let G be a p-divisible group over $\mathcal{O}_K$. If L is a finite extension of K, we have

$$\bigcap_{n=1}^{\infty} p^n G^\circ(\mathcal{O}_L) = 0.$$

PROOF. Since the valuation on L is discrete, there exists a minimum positive valuation δ on $\mathcal{O}_L$ given by the valuation of the uniformizer. Hence we find $p^n G^\circ(\mathcal{O}_L) \subseteq \mathrm{Fil}^{n\delta} G^\circ(\mathcal{O}_L)$ for each $n \geq 1$ by Lemma 3.2.14 and in turn deduce the desired assertion from Lemma 3.2.13. □

PROPOSITION 3.2.16. Let G be a p-divisible group over $\mathcal{O}_K$ and write $\mathscr{I}$ for the augmentation ideal of μ_G. There exists a map $\log_G : G(\mathcal{O}_L) \to t_G(L)$ with

$$\log_G(g)(\overline{\alpha}) = \lim_{n\to\infty} \frac{(p^n g)(\alpha)}{p^n} \quad \text{for every } g \in G(\mathcal{O}_L) \text{ and } \alpha \in \mathscr{I},$$

where $\overline{\alpha}$ denotes the image of α in $\mathscr{I}/\mathscr{I}^2$.

PROOF. Let us take arbitrary elements $g \in G(\mathcal{O}_L)$ and $\alpha \in \mathscr{I}$. We have $p^n g \in G^\circ(\mathcal{O}_L)$ for each $n \gg 0$ as noted in Proposition 3.2.9. Therefore Lemma 3.2.14 implies that there exists $c \in \mathbb{R}$ with $p^n g \in \mathrm{Fil}^{n+c} G^\circ(\mathcal{O}_L)$ for each $n \gg 0$ and in turn yields the inequality

$$\nu\left(\frac{(p^n g)(\beta)}{p^n}\right) \geq 2(n+c) - n = n + 2c \quad \text{for each } \beta \in \mathscr{I}^2. \tag{3.3}$$

Meanwhile, for each $n \gg 0$ we find

$$\frac{(p^{n+1}g)(\alpha)}{p^{n+1}} - \frac{(p^n g)(\alpha)}{p^n} = \frac{(p^n g)([p]_{\mu_G}(\alpha))}{p^{n+1}} - \frac{(p^n g)(\alpha)}{p^n} = \frac{(p^n g)([p]_{\mu_G}(\alpha) - p\alpha)}{p^{n+1}}.$$

Since we have $[p]_{\mu_G}(\alpha) - p\alpha \in \mathscr{I}^2$ by Lemma 2.2.13, we deduce from the inequality (3.3) that the sequence $\left(\frac{(p^n g)(\alpha)}{p^n}\right)$ converges in L. Moreover, if α lies in $\mathscr{I}^2$ the inequality (3.3) shows that the sequence converges to 0. The desired assertion is now evident. $\square$

Definition 3.2.17. Given a p-divisible group G over $\mathcal{O}_K$, we refer to the map $\log_G$ given by Proposition 3.2.16 as the *logarithm* of G.

Example 3.2.18. Let us provide an explicit description of $\log_{\mu_{p^\infty}}$. Choose isomorphisms

$$\mu_{p^\infty}(\mathcal{O}_L) \simeq \mathrm{Hom}_{\mathcal{O}_K-\mathrm{cont}}(\mathcal{O}_L[[t]], \mathcal{O}_L) \quad \text{and} \quad t_{\mu_{p^\infty}}(L) \simeq L$$

respectively given by Proposition 3.2.6 and Proposition 3.2.11. Since we have $\mu_{\widehat{\mathbb{G}_m}}[p^\infty] \cong \mu_{p^\infty}$ as noted in Example 2.2.12, for each $g \in \mu_{p^\infty}(\mathcal{O}_L)$ we find

$$(p^n g)(t) = g\left((1+t)^{p^n} - 1\right) = (1 + g(t))^{p^n} - 1.$$

Meanwhile, under the identification $\mu_{p^\infty}(\mathcal{O}_L) \cong 1 + \mathfrak{m}_L$ noted in Example 3.2.3, we identify each $g \in \mu_{p^\infty}(\mathcal{O}_L)$ with $1 + g(t)$. Hence obtain the identity

$$\log_{\mu_{p^\infty}}(1+x) = \lim_{n\to\infty} \frac{(1+x)^{p^n} - 1}{p^n} = \lim_{n\to\infty} \sum_{i=1}^{p^n} \frac{1}{p^n}\binom{p^n}{i} x^i \quad \text{for each } x \in \mathfrak{m}_L.$$

Moreover, for integers i and n we have

$$\frac{1}{p^n}\binom{p^n}{i} - \frac{(-1)^{i-1}}{i} = \frac{(p^n-1)\cdots(p^n-i+1) - (-1)^{i-1}(i-1)!}{i!}.$$

We observe that the numerator is divisible by p^n and in turn find

$$\nu\left(\frac{1}{p^n}\binom{p^n}{i} - \frac{(-1)^{i-1}}{i}\right) \geq n - \nu(i!) \geq n - \sum_{j=1}^{\infty} \frac{i}{p^j} = n - \frac{i}{p-1}.$$

Hence we obtain the expression

$$\log_{\mu_{p^\infty}}(1+x) = \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i} x^i \quad \text{for each } x \in \mathfrak{m}_L,$$

which coincides with the p-adic logarithm.

Let us state the following technical result about the logarithm maps without a proof.

PROPOSITION 3.2.19. Let G be a p-divisible group over $\mathcal{O}_K$ and write $\mathscr{I}$ for the augmentation ideal of μ_G. The map $\log_G$ is a local homeomorphism which induces an isomorphism

$$\mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) \simeq \left\{ \tau \in t_G(L) : \nu(\tau(f)) \geq \lambda \text{ for each } f \in \mathscr{I}/\mathscr{I}^2 \right\} \qquad \text{for every } \lambda \geq 1.$$

Remark. A key fact for the proof of Proposition 3.2.19 is that the multiplication by p on the group $G^\circ(\mathcal{O}_L)$ induces an isomorphism $\mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L) \cong \mathrm{Fil}^{\lambda+1} G^\circ(\mathcal{O}_L)$ as stated in the book of Serre [**Ser92**, Theorem 9.4]. It turns out that $\log_G$ admits a local inverse $\exp_G^\lambda$ on

$$\mathrm{Fil}^\lambda t_G(L) := \left\{ \tau \in t_G(L) : \nu(\tau(f)) \geq \lambda \text{ for each } f \in \mathscr{I}/\mathscr{I}^2 \right\}.$$

In fact, for every $\tau \in \mathrm{Fil}^\lambda t_G(L)$ we have $\exp_G^\lambda(\tau)(t_i) = \lim\limits_{n\to\infty} g_n(t_i)$ with each $g_n \in \mathrm{Fil}^\lambda G^\circ(\mathcal{O}_L)$ determined by the relation $(p^n g_n)(t_i) = p^n \tau(t_i)$.

PROPOSITION 3.2.20. Let G be a p-divisible group over $\mathcal{O}_K$ and denote by $\mathscr{I}$ the augmentation ideal of μ_G.

(1) $\log_G$ is a $\mathbb{Z}_p$-linear homomorphism.
(2) The kernel of $\log_G$ is the torsion subgroup $G(\mathcal{O}_L)_{\mathrm{tors}}$ of $G(\mathcal{O}_L)$.
(3) $\log_G$ induces a $\mathbb{Q}_p$-linear isomorphism $G(\mathcal{O}_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq t_G(L)$.

PROOF. Let us write $\mathscr{A}^\circ := \mathcal{O}_K[[t_1, \cdots, t_d]]$ where d is the dimension of G. Take arbitrary elements $g,\ h \in G(\mathcal{O}_L)$ and $\alpha \in \mathscr{I}$. We have $p^n g,\ p^n h \in G^\circ(\mathcal{O}_L)$ for each $n \gg 0$ as noted in Proposition 3.2.9. Since the axioms for μ_G yield the relation

$$\mu_G(\alpha) \in 1 \otimes \alpha + \alpha \otimes 1 + (\mathscr{I} \widehat{\otimes}_{\mathscr{A}^\circ} \mathscr{I})^2,$$

for each $n \gg 0$ we may write

$$(p^n(g+h))(\alpha) = (p^n g \otimes p^n h) \circ \mu_G(\alpha) = (p^n g)(\alpha) + (p^n h)(\alpha) + \beta_n$$

with $\beta_n \in (p^n g)(\mathscr{I}) \cdot (p^n h)(\mathscr{I})$. Moreover, we deduce from Lemma 3.2.14 that there exists a real number c with $p^n g,\ p^n h \in \mathrm{Fil}^{n+c} G^\circ(\mathcal{O}_L)$ for each $n \gg 0$ and in turn find $\nu(\beta_n) \geq 2(n+c)$. Now we obtain the identity

$$\lim_{n\to\infty} \frac{(p^n(g+h))(\alpha)}{p^n} = \lim_{n\to\infty} \frac{(p^n g)(\alpha)}{p^n} + \lim_{n\to\infty} \frac{(p^n h)(\alpha)}{p^n}$$

and consequently establish statement (1) by Proposition 3.2.19.

For statement (2), we only need to show that $\ker(\log_G)$ lies in $G(\mathcal{O}_L)_{\mathrm{tors}}$; indeed, we have $G(\mathcal{O}_L)_{\mathrm{tors}} \subseteq \ker(\log_G)$ as $t_G(L)$ is torsion free for being a vector space over L. Let us take an arbitrary element $g \in \ker(\log_G)$. Proposition 3.2.9 and Lemma 3.2.14 together imply that we have $p^n g \in \mathrm{Fil}^1 G^\circ(\mathcal{O}_L)$ for some $n \gg 0$. Since $p^n g$ lies in $\ker(\log_G)$ by statement (1), it must vanish by Proposition 3.2.19. We deduce that g is a torsion element and thus obtain statement (2).

Statement (2) readily implies that $\log_G$ induces an injective map $G(\mathcal{O}_L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to t_G(L)$. Moreover, we observe by Proposition 3.2.19 that this map is surjective as for each $\tau \in t_G(L)$ there exists an integer n with $p^n \tau \in \mathrm{Fil}^1 t_G(L)$. Hence we establish statement (3), thereby completing the proof. ∎

Remark. For $G = \mu_{p^\infty}$ and $L = \mathbb{C}_K$, the map $\log_{\mu_{p^\infty}}$ naturally extends to a Γ_K-equivariant group homomorphism $\log_p : \mathbb{C}_K^\times \to \mathbb{C}_K$ with $\log_p(p) = 0$, called the *Iwasawa logarithm*.

3.3. Hodge-Tate decomposition for Tate modules

In this subsection, we establish the main result for this chapter by exploiting our accumulated knowledge of finite flat group schemes and p-divisible groups.

LEMMA 3.3.1. Every p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$ yields canonical isomorphisms

$$G_v(\overline{K}) \cong G_v(\mathbb{C}_K) \cong G_v(\mathcal{O}_{\mathbb{C}_K}) \quad \text{for each } v \geq 1.$$

PROOF. Since the generic fiber of each G_v is finite étale as easily seen by Corollary 1.3.10, the first isomorphism follows from Proposition 3.1.10 and a standard fact stated in the Stacks project [**Sta**, Tag 0BND]. The second isomorphism is evident by the valuative criterion. □

LEMMA 3.3.2. Let G be a p-divisible group over $\mathcal{O}_K$.

(1) $G(\mathcal{O}_{\mathbb{C}_K})$ and $t_G(\mathbb{C}_K)$ admit natural Γ_K-actions with canonical identifications

$$G(\mathcal{O}_{\mathbb{C}_K})^{\Gamma_K} \cong G(\mathcal{O}_K) \quad \text{and} \quad t_G(\mathbb{C}_K)^{\Gamma_K} \cong t_G(K).$$

(2) The map $\log_G : G(\mathcal{O}_{\mathbb{C}_K}) \to t_G(\mathbb{C}_K)$ is Γ_K-equivariant.

PROOF. The group Γ_K naturally acts on $G(\mathcal{O}_{\mathbb{C}_K})$ by Proposition 3.2.5 and on $t_G(\mathbb{C}_K)$ by construction. Hence statement (1) follows from the identities $\mathbb{C}_K^{\Gamma_K} = K$ and $\mathcal{O}_{\mathbb{C}_K}^{\Gamma_K} = \mathcal{O}_K$ given by Theorem 3.1.11. Statement (2) is straightforward to verify. □

Definition 3.3.3. Let $G = \varinjlim G_v$ be a p-divisible group over $\mathcal{O}_K$.

(1) The *Tate module* of G is $T_p(G) := T_p(G \times_{\mathcal{O}_K} K) = \varprojlim G_v(\overline{K})$.

(2) The *Tate comodule* of G is $\Phi_p(G) := \varinjlim G_v(\overline{K})$.

Example 3.3.4. We have $T_p(\mu_{p^\infty}) = \mathbb{Z}_p(1)$ by definition and identify $\Phi_p(\mu_{p^\infty}) = \varinjlim \mu_{p^v}(\overline{K})$ with the group of p-power roots of unity in $\overline{K}$.

LEMMA 3.3.5. Let G be a p-divisible group G of height h over $\mathcal{O}_K$.

(1) $T_p(G)$ is a free $\mathbb{Z}_p$-module of rank h with a natural continuous Γ_K-action.

(2) $\Phi_p(G)$ is a torsion $\mathbb{Z}_p$-module with a natural continuous Γ_K-action.

PROOF. Let us write $G = \varinjlim G_v$ where each G_v is a finite flat $\mathcal{O}_K$-group. Corollary 1.3.10 shows that the generic fiber of each G_v is finite étale. Hence we deduce from Proposition 1.3.4 that each $G_v(\overline{K})$ is a free module of rank h over $\mathbb{Z}/p^v\mathbb{Z}$ with a natural continuous Γ_K-action and in turn establish the desired assertions. □

LEMMA 3.3.6. Every p-divisible group G over $\mathcal{O}_K$ naturally gives rise to a short exact sequence

$$0 \longrightarrow T_p((G^{\text{ét}})^\vee) \longrightarrow T_p(G^\vee) \longrightarrow T_p((G^\circ)^\vee) \longrightarrow 0.$$

PROOF. Let us write $G = \varinjlim G_v$ where each G_v is a finite flat $\mathcal{O}_K$-group. For each $v \geq 1$, Proposition 1.2.13 and Lemma 2.1.6 together yield a natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (G_{v+1}^{\text{ét}})^\vee(\overline{K}) & \longrightarrow & G_{v+1}^\vee(\overline{K}) & \longrightarrow & (G_{v+1}^\circ)^\vee(\overline{K}) & \longrightarrow & 0 \\ & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \downarrow\!\!\!\downarrow & & \\ 0 & \longrightarrow & (G_v^{\text{ét}})^\vee(\overline{K}) & \longrightarrow & G_v^\vee(\overline{K}) & \longrightarrow & (G_v^\circ)^\vee(\overline{K}) & \longrightarrow & 0 \end{array}$$

where both rows are exact. Therefore we obtain the desired exact sequence by a general fact stated in the Stacks project [**Sta**, Tag 03CA]. □

PROPOSITION 3.3.7. Given a p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$, there exist canonical Γ_K-equivariant $\mathbb{Z}_p$-module isomorphisms

$$T_p(G) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)) \quad \text{and} \quad \Phi_p(G) \cong \mathrm{Hom}_{\mathbb{Z}_p}\left(T_p(G^\vee), \Phi_p(\mu_{p^\infty})\right).$$

PROOF. Corollary 1.3.10 implies that the generic fiber of each G_v is finite étale. Hence each G_v gives rise to a canonical identification

$$G_v(\overline{K}) \cong (G_v^\vee)^\vee(\overline{K}) = \mathrm{Hom}_{\overline{K}\text{-grp}}\left((G_v^\vee)_{\overline{K}}, (\mu_{p^v})_{\overline{K}}\right) \cong \mathrm{Hom}(G_v^\vee(\overline{K}), \mu_{p^v}(\overline{K})) \tag{3.4}$$

by Theorem 1.2.4, Lemma 1.2.3, and Proposition 1.3.4. We deduce that $T_p(G)$ admits a natural Γ_K-equivariant isomorphism

$$\begin{aligned} T_p(G) &= \varprojlim G_v(\overline{K}) \cong \varprojlim \mathrm{Hom}(G_v^\vee(\overline{K}), \mu_{p^v}(\overline{K})) \\ &= \mathrm{Hom}_{\mathbb{Z}_p}(\varprojlim G_v^\vee(\overline{K}), \varprojlim \mu_{p^v}(\overline{K})) = \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)). \end{aligned}$$

Moreover, under the isomorphism $\Phi_p(G) = \varinjlim G_v(\overline{K}) \cong \varinjlim \mathrm{Hom}_{\mathbb{Z}_p}(G_v^\vee(\overline{K}), \Phi_p(\mu_{p^\infty}))$ given by the identification (3.4), we have a natural Γ_K-equivariant map

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \Phi_p(\mu_{p^\infty})) = \mathrm{Hom}_{\mathbb{Z}_p}(\varprojlim G_v^\vee(\overline{K}), \Phi_p(\mu_{p^\infty})) \longrightarrow \Phi_p(G)$$

which we verify to be an isomorphism using Lemma 2.1.6. □

PROPOSITION 3.3.8. Every p-divisible group $G = \varinjlim G_v$ over $\mathcal{O}_K$ yields a short exact sequence

$$0 \longrightarrow \Phi_p(G) \longrightarrow G(\mathcal{O}_{\mathbb{C}_K}) \xrightarrow{\log_G} t_G(\mathbb{C}_K) \longrightarrow 0.$$

PROOF. Since $G(\mathcal{O}_{\mathbb{C}_K})$ is p-divisible by Proposition 3.1.10 and Proposition 3.2.9, we deduce from Proposition 3.2.20 that $\log_G$ is surjective. In addition, we have

$$\ker(\log_G) = G(\mathcal{O}_{\mathbb{C}_K})_{\mathrm{tors}} \cong \varinjlim_v \varprojlim_i G_v(\mathcal{O}_{\mathbb{C}_K}/\mathfrak{m}^i\mathcal{O}_{\mathbb{C}_K}) = \varinjlim_v G_v(\mathcal{O}_{\mathbb{C}_K}) \cong \varinjlim_v G_v(\overline{K}) = \Phi_p(G)$$

by Proposition 3.2.20, Proposition 3.2.4, Lemma 3.2.1, and Lemma 3.3.1. □

LEMMA 3.3.9. Every p-divisible group G over $\mathcal{O}_K$ yields Γ_K-equivariant $\mathbb{Z}_p$-linear maps

$$\alpha : G(\mathcal{O}_{\mathbb{C}_K}) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1 + \mathfrak{m}_{\mathbb{C}_K}) \quad \text{and} \quad d\alpha : t_G(\mathbb{C}_K) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$$

via a natural isomorphism $T_p(G^\vee) \cong \mathrm{Hom}_{p\text{-div grp}}\left(G_{\mathcal{O}_{\mathbb{C}_K}}, (\mu_{p^\infty})_{\mathcal{O}_{\mathbb{C}_K}}\right)$.

PROOF. Let us write $G = \varinjlim G_v$ where each G_v is a finite flat $\mathcal{O}_K$-group. Lemma 3.3.1 and Lemma 1.2.3 together yield a canonical identification

$$\begin{aligned} T_p(G^\vee) &= \varprojlim G_v^\vee(\overline{K}) \cong \varprojlim G_v^\vee(\mathcal{O}_{\mathbb{C}_K}) \\ &= \varprojlim \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}_K}\text{-grp}}\left((G_v)_{\mathcal{O}_{\mathbb{C}_K}}, (\mu_{p^v})_{\mathcal{O}_{\mathbb{C}_K}}\right) \\ &= \mathrm{Hom}_{p\text{-div grp}}\left(G_{\mathcal{O}_{\mathbb{C}_K}}, (\mu_{p^\infty})_{\mathcal{O}_{\mathbb{C}_K}}\right). \end{aligned}$$

In addition, we have $\mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K}) \cong 1 + \mathfrak{m}_{\mathbb{C}_K}$ and $t_{\mu_{p^\infty}}(\mathbb{C}_K) \cong \mathbb{C}_K$ as noted in Example 3.2.18. Hence each $w \in T_p(G^\vee)$ gives rise to maps

$$w_{\mathcal{O}_{\mathbb{C}_K}} : G(\mathcal{O}_{\mathbb{C}_K}) \longrightarrow \mu_{p^\infty}(\mathcal{O}_{\mathbb{C}_K}) \cong 1 + \mathfrak{m}_{\mathbb{C}_K} \quad \text{and} \quad dw_{\mathbb{C}_K} : t_G(\mathbb{C}_K) \longrightarrow t_{\mu_{p^\infty}}(\mathbb{C}_K) \cong \mathbb{C}_K.$$

Now we obtain the desired maps α and $d\alpha$ by setting

$$\alpha(g)(w) := w_{\mathcal{O}_{\mathbb{C}_K}}(g) \quad \text{and} \quad d\alpha(\tau)(w) := dw_{\mathbb{C}_K}(\tau)$$

for each $g \in G(\mathcal{O}_{\mathbb{C}_K})$, $\tau \in t_G(\mathbb{C}_K)$, and $w \in T_p(G^\vee)$. □

Proposition 3.3.10. Given a p-divisible group G over $\mathcal{O}_K$, there exists a canonical commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Phi_p(G) & \longrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\log_G} & t_G(\mathbb{C}_K) & \longrightarrow & 0 \\ & & \downarrow\wr & & \downarrow\alpha & & \downarrow d\alpha & & \\ 0 & \to & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \Phi_p(\mu_{p^\infty})) & \to & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K}) & \to & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) & \to & 0 \end{array}$$

with exact rows and Γ_K-equivariant arrows.

Proof. The vertical arrows in the diagram are the natural Γ_K-equivariant maps given by Proposition 3.3.7 and Lemma 3.3.9. The horizontal arrows in the diagram are the natural maps given by Proposition 3.3.8 and are Γ_K-equivariant by Lemma 3.3.2. We observe that both rows are exact by Lemma 3.3.5 and Proposition 3.3.8. In addition, it is straightforward to verify that the diagram is commutative. Hence it remains to prove that α and $d\alpha$ are injective. We only need to show that $d\alpha$ is injective as we have $\ker(\alpha) \simeq \ker(d\alpha)$ by the snake lemma.

We assert that α is injective on $G(\mathcal{O}_K)$. Suppose for contradiction that there exists a nonzero element $g \in \ker(\alpha)$. The $\mathbb{Z}_p$-linear map $d\alpha$ is in fact $\mathbb{Q}_p$-linear as both $t_G(\mathbb{C}_K)$ and $\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$ are vector spaces over $\mathbb{Q}_p$. We deduce that $\ker(\alpha) \simeq \ker(d\alpha)$ is also a vector space over $\mathbb{Q}_p$ and thus is torsion free. Now we may assume by Proposition 3.2.9 that g lies in $G^\circ(\mathcal{O}_K)$. Lemma 3.3.9 yields a commutative diagram

$$\begin{array}{ccc} G^\circ(\mathcal{O}_{\mathbb{C}_K}) & \hookrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) \\ \downarrow\alpha^\circ & & \downarrow\alpha \\ \mathrm{Hom}_{\mathbb{Z}_p}(T_p((G^\circ)^\vee), 1+\mathfrak{m}_{\mathbb{C}_K}) & \hookrightarrow & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K}) \end{array}$$

where the injectivity of the horizontal maps follow from Proposition 3.2.8 and Lemma 3.3.6. We find $g \in \ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K)$ and obtain the identification $\ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K) = \ker(\alpha^\circ)^{\Gamma_K}$ by Lemma 3.3.2. Since $\ker(\alpha^\circ)^{\Gamma_K}$ is a vector space over $\mathbb{Q}_p$, for every integer $n \geq 0$ there exists an element $g_n \in \ker(\alpha^\circ) \cap G^\circ(\mathcal{O}_K)$ with $g = p^n g_n$. We deduce from Lemma 3.2.15 that g must be zero and in turn obtain a desired contradiction.

Now we show that $d\alpha$ is injective on $t_G(K)$. It is enough to establish the injectivity on $\log_G(G(\mathcal{O}_K))$ as we have $\log_G(G(\mathcal{O}_K)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = t_G(K)$ by Proposition 3.2.20. Let us take an arbitrary element $h \in G(\mathcal{O}_K)$ with $\log_G(h) \in \ker(d\alpha)$. Since $\log_G$ induces the isomorphism $\ker(\alpha) \simeq \ker(d\alpha)$ by the snake lemma, we find $\log_G(h) = \log_G(h')$ for some $h' \in \ker(\alpha)$. Proposition 3.2.20 implies that $h - h'$ is torsion, which means that there exists $n \geq 0$ with $p^n(h - h') = 0$ or equivalently $p^n h = p^n h'$. Hence we have $p^n h \in \ker(\alpha) \cap G(\mathcal{O}_K)$ and in turn find $p^n h = 0$ by the injectivity of α on $G(\mathcal{O}_K)$. We deduce from Proposition 3.2.20 that $\log_G(h)$ is zero, which implies that $d\alpha$ is injective on $\log_G(G(\mathcal{O}_K))$.

Our discussion in the previous paragraph shows that $d\alpha$ factors through an injective map

$$t_G(\mathbb{C}_K) \cong t_G(K) \otimes_K \mathbb{C}_K \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K} \otimes_K \mathbb{C}_K.$$

In addition, Proposition 3.1.12 yields an injective map

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K} \otimes_K \mathbb{C}_K \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), K) \otimes_K \mathbb{C}_K \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$$

where the isomorphism comes from the fact that $T_p(G^\vee)$ is free over $\mathbb{Z}_p$ by Lemma 3.3.5. Now we identify $d\alpha$ with the composition of these maps and in turn establish its injectivity, thereby completing the proof. $\square$

THEOREM 3.3.11 (Tate [**Tat67**]). Let G be a p-divisible group over $\mathcal{O}_K$.

(1) There exist natural isomorphisms
$$G(\mathcal{O}_K) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K})^{\Gamma_K} \quad \text{and} \quad t_G(K) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K}.$$

(2) The tangent spaces $t_G(\mathbb{C}_K)$ and $t_{G^\vee}(\mathbb{C}_K)$ are orthogonal complements with respect to a $\mathbb{C}_K$-linear Γ_K-equivariant perfect pairing
$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \times \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) \to \mathbb{C}_K(-1).$$

PROOF. Proposition 3.3.10 and the snake lemma together yield a commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(\mathcal{O}_{\mathbb{C}_K}) & \xrightarrow{\alpha} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K}) & \longrightarrow & \mathrm{coker}(\alpha) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \log_G} & & \big\downarrow & & \big\downarrow{\scriptstyle \wr} & & \\ 0 & \longrightarrow & t_G(\mathbb{C}_K) & \xrightarrow{d\alpha} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) & \longrightarrow & \mathrm{coker}(d\alpha) & \longrightarrow & 0 \end{array}$$
where both rows are exact. We apply Lemma 3.3.2 to obtain a commutative diagram
$$\begin{array}{ccccccc} 0 & \longrightarrow & G(\mathcal{O}_K) & \xrightarrow{\alpha_K} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), 1+\mathfrak{m}_{\mathbb{C}_K})^{\Gamma_K} & \longrightarrow & \mathrm{coker}(\alpha)^{\Gamma_K} \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \wr} \\ 0 & \longrightarrow & t_G(K) & \xrightarrow{d\alpha_K} & \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K} & \longrightarrow & \mathrm{coker}(d\alpha)^{\Gamma_K} \end{array}$$
where both rows are exact. We observe that the middle vertical map induces an injective map
$$\mathrm{coker}(\alpha_K) \hookrightarrow \mathrm{coker}(d\alpha_K). \tag{3.5}$$
In addition, we switch the roles of G and $G^\vee$ to get an injective map
$$d\alpha_K^\vee : t_{G^\vee}(K) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K)^{\Gamma_K}.$$

Let us denote the height of G by h. Proposition 2.1.9 and Lemma 3.3.5 together show that $V := \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K)$ and $W := \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$ are vector spaces over $\mathbb{C}_K$ of dimension h. Moreover, Proposition 3.3.7 yields a Γ_K-equivariant $\mathbb{Z}_p$-linear perfect pairing
$$T_p(G) \times T_p(G^\vee) \to \mathbb{Z}_p(1),$$
which in turn gives rise to a Γ_K-equivariant $\mathbb{C}_K$-linear perfect pairing
$$V \times W \to \mathbb{C}_K(-1). \tag{3.6}$$
This pairing maps $V^{\Gamma_K} \times W^{\Gamma_K}$ into $\mathbb{C}_K(-1)^{\Gamma_K}$, which is zero by Theorem 3.1.11. We deduce that $V^{\Gamma_K} \otimes_K \mathbb{C}_K$ and $W^{\Gamma_K} \otimes_K \mathbb{C}_K$ are orthogonal and consequently find
$$\dim_K(V^{\Gamma_K}) + \dim_K(W^{\Gamma_K}) \leq \dim_{\mathbb{C}_K}(V) = h.$$
Meanwhile, the injectivity of $d\alpha_K$ and $d\alpha_K^\vee$ yields the inequality
$$\dim_K(V^{\Gamma_K}) + \dim_K(W^{\Gamma_K}) \geq \dim_K(t_G(K)) + \dim_K(t_{G^\vee}(K)) = h$$
where the equality follows from Theorem 2.2.23 and Proposition 3.2.11. Therefore all inequalities are in fact equalities. We deduce that the injective map $d\alpha_K$ is an isomorphism and in turn find by the injective map (3.5) that α_K is also an isomorphism. Now we establish statement (1), which in particular yields natural identifications
$$t_G(\mathbb{C}_K) \cong W^{\Gamma_K} \otimes_K \mathbb{C}_K \quad \text{and} \quad t_{G^\vee}(\mathbb{C}_K) \cong V^{\Gamma_K} \otimes_K \mathbb{C}_K.$$
Our discussion readily shows that these spaces are orthogonal under the pairing (3.6) with $\dim_{\mathbb{C}_K}(t_G(\mathbb{C}_K)) + \dim_{\mathbb{C}_K}(t_{G^\vee}(\mathbb{C}_K)) = \dim_{\mathbb{C}_K}(V)$, thereby implying statement (2). □

Proposition 3.3.12. Given a p-divisible group G of dimension d over $\mathcal{O}_K$, we have

$$d = \dim_K \left(\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)^{\Gamma_K}\right) = \dim_K (T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1))^{\Gamma_K}.$$

Proof. The first equality is evident by Proposition 3.2.11 and Theorem 3.3.11. The second equality follows from the identification

$$T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{Z}_p(1)) \otimes_{\mathbb{Z}_p} \mathbb{C}_K(-1) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K)$$

given by Lemma 3.3.5 and Proposition 3.3.7. □

Remark. Lemma 3.3.5 and Proposition 3.3.12 together show that we can compute the height and the dimension of G from $T_p(G)$.

Theorem 3.3.13 (Tate [**Tat67**]). Every p-divisible group G over $\mathcal{O}_K$ gives rise to a canonical $\mathbb{C}_K[\Gamma_K]$-module isomorphism

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \cong t_{G^\vee}(\mathbb{C}_K) \oplus t_G^*(\mathbb{C}_K)(-1).$$

Proof. We identify $t_G^*(\mathbb{C}_K)$ with the $\mathbb{C}_K$-dual $t_G(\mathbb{C}_K)$ and find

$$\mathrm{Hom}_{\mathbb{C}_K}(t_G(\mathbb{C}_K), \mathbb{C}_K(-1)) \cong t_G^*(\mathbb{C}_K)(-1).$$

Since Theorem 3.3.11 yields a $\mathbb{C}_K$-linear Γ_K-equivariant perfect pairing

$$\mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \times \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G^\vee), \mathbb{C}_K) \to \mathbb{C}_K(-1)$$

under which $t_G(\mathbb{C}_K)$ and $t_{G^\vee}(\mathbb{C}_K)$ are orthogonal complements, we get a short exact sequence

$$0 \longrightarrow t_{G^\vee}(\mathbb{C}_K) \longrightarrow \mathrm{Hom}_{\mathbb{Z}_p}(T_p(G), \mathbb{C}_K) \longrightarrow t_G^*(\mathbb{C}_K)(-1) \longrightarrow 0 \tag{3.7}$$

where all maps are $\mathbb{C}_K$-linear and Γ_K-equivariant. Let us write $d := \dim_{\mathbb{C}_K}(t_G(\mathbb{C}_K))$ and $d^\vee := \dim_{\mathbb{C}_K}(t_{G^\vee}(\mathbb{C}_K))$. We have isomorphisms

$$\mathrm{Ext}^1_{\mathbb{C}_K[\Gamma_K]}(t_G^*(\mathbb{C}_K)(-1), t_{G^\vee}(\mathbb{C}_K)) \simeq \mathrm{Ext}^1_{\mathbb{C}_K[\Gamma_K]}(\mathbb{C}_K(-1)^{\oplus d^\vee}, \mathbb{C}_K^{\oplus d}) \simeq H^1(\Gamma_K, \mathbb{C}_K(1))^{\oplus dd^\vee},$$

$$\mathrm{Hom}_{\mathbb{C}_K[\Gamma_K]}(t_G^*(\mathbb{C}_K)(-1), t_{G^\vee}(\mathbb{C}_K)) \simeq \mathrm{Hom}_{\mathbb{C}_K[\Gamma_K]}(\mathbb{C}_K(-1)^{\oplus d^\vee}, \mathbb{C}_K^{\oplus d}) \simeq H^0(\Gamma_K, \mathbb{C}_K(1))^{\oplus dd^\vee}.$$

Theorem 3.1.11 shows that both $H^0(\Gamma_K, \mathbb{C}_K(1))$ and $H^1(\Gamma_K, \mathbb{C}_K(1))$ vanish. Hence we deduce that the exact sequence (3.7) canonically splits, thereby establishing the desired assertion. □

Definition 3.3.14. Given a p-divisible group G over $\mathcal{O}_K$, we refer to the isomorphism in Theorem 3.3.13 as the *Hodge-Tate decomposition* for G.

Corollary 3.3.15. For every p-divisible group G over $\mathcal{O}_K$, the rational Tate-module

$$V_p(G) = T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a Hodge-Tate p-adic Γ_K-representation.

Proof. Let us identify the $\mathbb{C}_K$-duals of $t_{G^\vee}(\mathbb{C}_K)$ and $t_G^*(\mathbb{C}_K)$ respectively with $t_{G^\vee}^*(\mathbb{C}_K)$ and $t_G(\mathbb{C}_K)$. Theorem 3.3.13 yields a natural decomposition

$$V_p(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong t_{G^\vee}^*(\mathbb{C}_K) \oplus t_G(\mathbb{C}_K)(1).$$

Therefore we apply Theorem 3.1.11 to find

$$\left(V_p(G) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(-n)\right)^{\Gamma_K} \cong \begin{cases} t_{G^\vee}^*(K) & \text{for } n = 0, \\ t_G(K) & \text{for } n = 1, \\ 0 & \text{for } n \neq 0,\ 1. \end{cases}$$

The desired assertion is now evident. □

Remark. Our proof of Corollary 3.3.15 shows that we can find $t_G(K)$ from $T_p(G)$.

Proposition 3.3.16. Let A be an abelian variety over K.

(1) There exists a canonical isomorphism

$$H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \cong \mathrm{Hom}_{\mathbb{Z}_p}(T_p(A[p^\infty]), \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

(2) If A has good reduction, its integral model $\mathcal{A}$ over $\mathcal{O}_K$ yields natural isomorphisms

$$H^0(A, \Omega^1_{A/K}) \cong t^*_{\mathcal{A}[p^\infty]}(K) \quad \text{and} \quad H^1(A, \mathcal{O}_A) \cong t_{\mathcal{A}^\vee[p^\infty]}(K).$$

(3) Given integers $i,\ j \geq 0$ and $n \geq 0$, we have natural identifications

$$\begin{aligned} H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) &\cong \bigwedge\nolimits^n H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p), \\ H^i(A, \Omega^j_{A/K}) &\cong \bigwedge\nolimits^i H^1(A, \mathcal{O}_A) \otimes \bigwedge\nolimits^j H^0(A, \Omega^1_{A/K}). \end{aligned}$$

Proof. All assertions are standard facts about abelian varieties stated in the notes of Milne [**Mil**, §7, §12] and the book of Mumford [**Mum70**, §4]. □

Theorem 3.3.17. Given an abelian variety A over K with good reduction, there exists have a canonical Γ_K-equivariant isomorphism

$$H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \bigoplus_{i+j=n} H^i(A, \Omega^j_{A/K}) \otimes_K \mathbb{C}_K(-j) \quad \text{for each } n \geq 1.$$

Proof. Since A has good reduction, it admits an integeral model $\mathcal{A}$ over $\mathcal{O}_K$. We have $T_p(\mathcal{A}[p^\infty]) = T_p(A[p^\infty])$ by definition and find $\mathcal{A}^\vee[p^\infty] \cong \mathcal{A}[p^\infty]^\vee$ by Example 2.1.11. Hence Theorem 3.3.13 and Proposition 3.3.16 together yield a canonical Γ_K-equivariant isomorphism

$$H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong (H^1(A, \mathcal{O}_A) \otimes_K \mathbb{C}_K) \oplus (H^0(A, \Omega^1_{A/K}) \otimes_K \mathbb{C}_K(-1)).$$

Now we deduce the desired assertion from Proposition 3.3.16. □

Remark. Theorem 3.3.17 is a special case of the Hodge-Tate decomposition theorem that we have introduced in Chapter I, Theorem 1.2.2. The proof of the Hodge-Tate decomposition theorem for the general case requires ideas that are beyond the scope of our discussion. We refer curious readers to the notes of Bhatt [**Bha19**] for a wonderful exposition of the proof by Scholze [**Sch13**] using his theory of *perfectoid spaces*.

Corollary 3.3.18. For every abelian variety A over K with good reduction, the étale cohomology $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ for each $n \geq 1$ is a Hodge-Tate p-adic Γ_K-representation.

Proof. Let us take an arbitary integer m. If we have $0 \leq m \leq n$, Theorem 3.1.11 and Theorem 3.3.17 together yield a natural isomorphism

$$\left(H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(m)\right)^{\Gamma_K} \cong H^{n-m}(A, \Omega^m_{A/K}).$$

Otherwise, Theorem 3.1.11 and Theorem 3.3.17 imply that $\left(H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(m)\right)^{\Gamma_K}$ is trivial. Now the desired assertion is straightforward to verify. □

Remark. In fact, given a proper smooth variety X over K, the Hodge-Tate decomposition theorem implies that the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ for each integer $n \geq 1$ is a Hodge-Tate p-adic Γ_K-representation.

Exercises

1. In this exercise, we study homomorphisms between the R-groups $\mathbb{G}_a$ and $\mathbb{G}_m$.
   (1) Show that every homomorphism from $\mathbb{G}_m$ to $\mathbb{G}_a$ is trivial.
   (2) If R is reduced, show that every homomorphism from $\mathbb{G}_a$ to $\mathbb{G}_m$ is trivial.
   (3) If R contains a nonzero element α with $\alpha^2 = 0$, construct a nonzero homomorphism from $\mathbb{G}_a$ to $\mathbb{G}_m$.

2. Assume that $R = k$ is a field.
   (1) Establish a canonical isomorphism $\mathrm{End}_{k\text{-grp}}(\mathbb{G}_m) \cong \mathbb{Z}$.
   (2) If k has characteristic 0, establish a natural identification $\mathrm{End}_{k\text{-grp}}(\mathbb{G}_a) \cong k$.
   (3) If k has characteristic p, show that $\mathrm{End}_{k\text{-grp}}(\mathbb{G}_a)$ is isomorphic to the (possibly noncommutative) polynomial ring $k\langle\varphi\rangle$ with $\varphi c = c^p\varphi$ for any $c \in k$.

3. Assume that $R = k$ is a field of characteristic p.
   (1) Show that the k-algebra homomorphism $k[t] \to k[t]$ which sends t to $t^p - t$ induces a k-group homomorphism $f : \mathbb{G}_a \to \mathbb{G}_a$.
   (2) Show that $\ker(f)$ is isomorphic to $\underline{\mathbb{Z}/p\mathbb{Z}}$.

4. Assume that $R = k$ is a field of characteristic p.
   (1) Verify that the k-group $\alpha_{p^2} := \mathrm{Spec}\,(k[t]/t^{p^2})$ with the natural additive group structure on $\alpha_{p^2}(B) = \left\{ b \in B : b^{p^2} = 0 \right\}$ for each k-algebra B is finite flat of order p^2.
   (2) Show that $\alpha_{p^2}^\vee$ admits an isomorphism $\alpha_{p^2}^\vee \cong \mathrm{Spec}\,(k[t,u]/(t^p, u^p))$ with the multiplication on $\alpha_{p^2}^\vee(B) \cong \{ (b_1, b_2) \in B^2 : b_1^p = b_2^p = 0 \}$ for each k-algebra B given by
   $$(b_1, b_2) \cdot (b_1', b_2') = \big(b_1 + b_1', b_2 + b_2' - W_1(b_1, b_2)\big)$$
   where we write $W_1(t,u) := \dfrac{(t+u)^p - t^p - u^p}{p} \in \mathbb{Z}[t,u]$.

   Hint. We can show that a B-algebra homomorphism $f : B[t, t^{-1}] \to B[t]/(t^{p^2})$ induces a B-group homomorphism $\alpha_{p^2} \to \mathbb{G}_m$ if and only if $f(t)$ admits an identity
   $$f(t) = \sum_{i=0}^{p-1} \frac{(b_1 t)^i}{i!} \sum_{j=0}^{p-1} \frac{(b_2 t^p)^j}{i!}$$
   for some $b_1, b_2 \in B$ with $b_1^p = b_2^p = 0$.
   (3) For $k = \overline{\mathbb{F}}_p$, show that α_{p^2} fits into a nonsplit short exact sequence
   $$\underline{0} \longrightarrow \alpha_p \longrightarrow \alpha_{p^2} \longrightarrow \alpha_p \longrightarrow \underline{0}.$$

Remark. For $k = \overline{\mathbb{F}}_p$, there exists a natural identification
$$\mathrm{Ext}^1_{\overline{\mathbb{F}}_p\text{-grp}}(\alpha_p, \alpha_p) \cong (\mathbb{Z}/2\mathbb{Z})^2$$
with elements given by α_p^2, α_{p^2}, $\alpha_{p^2}^\vee$, and the p-torsion part of a supersingular elliptic curve.

5. Assume that $R = k$ is a perfect field.

(1) Given a finite abelian group M with a continuous Γ_k-action, show that the scheme $\underline{M}^{\Gamma_k} := \text{Spec}\,(A)$ for $A := \Big(\prod_{i \in M} \overline{k}\Big)^{\Gamma_k}$ is naturally a finite étale k-group.

Hint. Since M is finite, the Γ_k-action should factor through a finite quotient.

(2) Prove that the inverse functor for the equivalence in Proposition 1.3.4 maps each finite abelian group M with a continuous Γ_k-action to $\underline{M}^{\Gamma_k}$.

(3) Prove that a finite étale group scheme G over a field k is a constant group scheme if and only if the Γ_k-action on $G(\overline{k})$ is trivial.

6. Given a nonperfect field k of characteristic p, let c be an element of k which is not a p-th power and set $G := \coprod_{i=0}^{p-1} G_i$ with $G_i := \text{Spec}\,\big(k[t]/(t^p - c^i)\big)$.

(1) Given a k-algebra B, verify a natural identification

$$G_i(B) \cong \big\{\, b \in B : b^p = c^i \,\big\} \qquad \text{for each } i = 0, \cdots, p-1$$

and show that $G(B)$ is a group with multiplication given by the following maps:

- $m_{ij} : G_i(B) \times G_j(B) \to G_{i+j}(B)$ for $i + j < p$ sending each (g, g') to gg',
- $m_{ij} : G_i(B) \times G_j(B) \to G_{i+j-p}(B)$ for $i + j \geq p$ sending each (g, g') to gg'/c.

(2) Show that G yields a nonsplit connected-étale sequence

$$\underline{0} \longrightarrow \mu_p \longrightarrow G \longrightarrow \underline{\mathbb{Z}/p\mathbb{Z}} \longrightarrow \underline{0}.$$

Hint. To show that the sequence does not split, compare G_0 with G_i for $i \neq 0$.

Remark. This exercise shows that Proposition 1.4.15 fails when the base field is not perfect. The notes of Pink [**Pin**, §15] provide an analogous example with $G_i = \text{Spec}\,(k[t]/(t^p - ic))$.

7. Assume that $R = k$ is a field.

(1) Give a proof of Theorem 1.3.9 when $R = k$ is a field without using Theorem 1.1.17.

Hint. If k has characteristic 0, we can adjust the proof of Proposition 1.5.19 to show that G° is trivial.

(2) Prove Theorem 1.1.17 when $R = k$ is a field.

Hint. If k has characteristic 0, we can deduce the assertion from Lagrange's theorem for finite groups by observing that G is étale. If k has characteristic p, we can reduce to the case where G is simple with k algebraically closed.

8. Let E be an elliptic curve over $\overline{\mathbb{F}}_p$.

(1) Show that E is either ordinary or supersingular.

(2) If E is supersingular, show that $\ker(\varphi_{E[p]})$ is isomorphic to α_p.

9. Assume that $R = k$ is a perfect field.
   (1) Show that the dual of every étale p-divisible group over k is connected.
   (2) Show that every p-divisible G over k admits a natural decomposition
   $$G \cong G^{\text{ll}} \times G^{\text{mult}} \times G^{\text{ét}}$$
   with the following properties:
      (i) G^{ll} is connected with $(G^{\text{ll}})^\vee$ being connected.
      (ii) G^{mult} is connected with $(G^{\text{mult}})^\vee$ being étale.
      (iii) $G^{\text{ét}}$ is étale with $(G^{\text{ét}})^\vee$ being connected.

10. Assume that $R = k$ is a field of characteristic 0. Establish an isomorphism between the formal group laws $\mu_{\widehat{\mathbb{G}_a}}$ and $\mu_{\widehat{\mathbb{G}_m}}$ over k.

Hint. Consider the map $k[[t]] \to k[[t]]$ sending t to $\exp(t) - 1 = \displaystyle\sum_{n=1}^{\infty} \frac{t^n}{n!}$.

11. Let K be a finite extension of $\mathbb{Q}_p$ with uniformizer π and residue field $\mathbb{F}_q$.
   (1) Show that there exists a unique formal group law μ_π over $\mathcal{O}_K$ of dimension 1 with an endomorphism $[\pi] : \mathcal{O}_K[[t]] \to \mathcal{O}_K[[t]]$ sending t to $\pi t + t^q$.
   (2) Show that μ_π is p-divisible.

Remark. The formal group law μ_π is a *Lubin-Tate formal group law*, introduced by the work of Lubin-Tate [**LT65**] as a means to construct the totally ramified abelian extensions of K.

12. If $R = k$ is an algebraically closed field of characteristic p, prove that every étale p-divisible group of height h over k is isomorphic to $(\underline{\mathbb{Q}_p/\mathbb{Z}_p})^h$.

13. In this exercise, we study the p-adic expansion and the Teichmüler expansion on $\mathbb{Z}_p$.
   (1) Show that the 2-adic expansion agrees with the Teichmüler expansion on $\mathbb{Z}_2$.
   (2) Show that the p-adic expansion does not agree with the Teichmüler expansion on $\mathbb{Z}_p$ for $p > 2$.
   (3) Find the 3-adic expansion for $[2] \in \mathbb{Z}_3$.
   (4) Find the first four coefficients of the 5-adic expansion for $[2] \in \mathbb{Z}_5$.

Hint. The Teichmüler lift of an element $a \in \mathbb{F}_p$ is the unique lift $[a] \in \mathbb{Z}_p$ with $[a]^p = [a]$. We can inductively find its image in $\mathbb{Z}_p/p^n\mathbb{Z}_p = \mathbb{Z}/p^n\mathbb{Z}$ for each $n \geq 1$ by Hensel's lemma.

14. Assume that $R = k$ is a perfect field of characteristic p and write $K_0(k) := W(k)[1/p]$.
   (1) For every $\lambda \in \mathbb{Q}$, establish a natural isomorphism of isocrystals $D_\lambda^\vee \cong D_{-\lambda}$.
   (2) For every $\lambda, \lambda' \in \mathbb{Q}$, establish a natural isomorphism of isocrystals
   $$D_\lambda \otimes_{K_0(k)} D_{\lambda'} \cong D_{\lambda+\lambda'}^{\oplus n} \quad \text{with } n \geq 1.$$

15. Let E be an arbitrary field.
   (1) Show that the p-adic Γ_E-representation $V_p(\underline{\mathbb{Q}_p/\mathbb{Z}_p})$ is trivial.
   (2) If E has characteristic p, show that the p-adic cyclotomic character χ_E is trivial.

16. Let K be a p-adic field.
   (1) Prove that the algebraic closure $\overline{K}$ of K is not p-adically complete.
   Hint. There are at least two ways to proceed as follows:
      (a) We can observe that $\overline{K}$ is a countable union of nowhere dense subspaces and apply the Baire category theorem to conclude.
      (b) Alternatively, we can use Krasner's lemma to produce a Cauchy sequence in $\overline{K}$ whose limit is not algebraic over K.
   (2) Given an extension L of K with $L \subseteq \overline{K}$, prove that its closure $\widehat{L}$ in $\mathbb{C}_K$ yields an identity $L = \widehat{L} \cap \overline{K}$.

17. In this exercise, we study the logarithm of μ_{p^∞} over $\mathcal{O}_K$ for a p-adic field K.
   (1) Give a proof of Proposition 3.2.19 for $G = \mu_{p^\infty}$.
   (2) Show that the map $\log_{\mu_{p^\infty}}$ naturally extends to a Γ_K-equivariant group homomorphism $\log_p : \mathbb{C}_K^\times \to \mathbb{C}_K$ with $\log_p(p) = 0$.

18. Let K be a p-adic field and G be a p-divisible group over $\mathcal{O}_K$.
   (1) Show that there exist canonical $\mathbb{Z}_p$-linear Γ_K-equivariant isomorphisms
   $$T_p(G) \cong \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p, \Phi_p(G)) \quad \text{and} \quad \Phi_p(G) \cong T_p(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p.$$
   (2) Given a p-divisible group H over $\mathcal{O}_K$, show that the generic fibers of G and H are isomorphic if and only if G and H satisfy the following equivalent conditions:
      (i) The $\mathbb{Z}_p[\Gamma_K]$-modules $T_p(G)$ and $T_p(H)$ are isomorphic.
      (ii) The $\mathbb{Z}_p[\Gamma_K]$-modules $\Phi_p(G)$ and $\Phi_p(H)$ are isomorphic.

19. Let K be a p-adic field and E be an elliptic curve over $\mathcal{O}_K$.
   (1) Prove that E gives rise to a natural Γ_K-equivariant $\mathbb{Z}_p$-linear perfect pairing
   $$T_p(E[p^\infty]) \times T_p(E[p^\infty]) \longrightarrow \mathbb{Z}_p(1).$$
   (2) Prove that the determinant character of the Γ_K-representation $V_p(E[p^\infty])$ coincides with the p-adic cyclotomic character.

Remark. The perfect pairing in the first part coincides with the *Weil pairing* on E.

20. Let K be a p-adic field and G be a p-divisible group over $\mathcal{O}_K$ with an exact sequence
$$0 \longrightarrow \mathbb{Q}_p(m) \longrightarrow V_p(G) \longrightarrow \mathbb{Q}_p(n) \longrightarrow 0.$$
   (1) Show that m and n respectively satisfy the inequalities $0 \leq m \leq 1$ and $0 \leq n \leq 1$.
   (2) Show that G is étale if and only if m and n satisfy the equality $m = n = 0$.
   (3) Show that G is connected if and only if m and n satisfy the equality $m = n = 1$.

CHAPTER III

Period rings and functors

1. Fontaine's formalism on period rings

The main goal of this section is to discuss the formalism developed by Fontaine [**Fon94a**] for p-adic period rings and their associated functors. Our primary references for this section are the notes of Brinon-Conrad [**BC**, §5] and the notes of Fontaine-Oiyang [**FO**, §2.1].

Throughout this chapter, we let K be a p-adic field with absolute Galois group Γ_K, inertia group I_K, and residue field k. In addition, we write $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ for the category of p-adic Γ_K-representations and χ for the p-adic cyclotomic character of K.

1.1. Basic definitions and examples

In this subsection, we define some key notions for our formalism and relate them to Hodge-Tate representations.

Definition 1.1.1. An integral domain B over $\mathbb{Q}_p$ with an action of Γ_K is $(\mathbb{Q}_p, \Gamma_K)$-*regular* if it satisfies the following conditions:

(i) We have $B^{\Gamma_K} = C^{\Gamma_K}$, where C denotes the fraction field of B endowed with a natural Γ_K-action extending the Γ_K-action on B.

(ii) A nonzero $b \in B$ is a unit if $\mathbb{Q}_p b = \{ cb \in B : c \in \mathbb{Q}_p \}$ is stable under the Γ_K-action.

Remark. For any field F and any group Γ, we can similarly define (F, Γ)-regular rings. The formalism that we develop in this section readily extends to (F, Γ)-regular rings.

Example 1.1.2. Every field extension of $\mathbb{Q}_p$ with an action of Γ_K is $(\mathbb{Q}_p, \Gamma_K)$-regular.

Definition 1.1.3. Let B be a $(\mathbb{Q}_p, \Gamma_K)$-regular ring with $E := B^{\Gamma_K}$.

(1) We define *the functor associated to* B to be $D_B : \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K) \to \mathrm{Vect}_E$ with

$$D_B(V) := (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \qquad \text{for every } V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K),$$

where Vect_E denotes the category of vector spaces over E.

(2) We say that $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ is B-*admissible* if it satisfies the equality

$$\dim_E D_B(V) = \dim_{\mathbb{Q}_p} V.$$

Example 1.1.4. For every $(\mathbb{Q}_p, \Gamma_K)$-regular ring B, trivial p-adic Γ_K-representations are B-admissible.

Definition 1.1.5. Given a character $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ and a $\mathbb{Q}_p[\Gamma_K]$-module M, we define the η-*twist of* M to be the $\mathbb{Q}_p[\Gamma_K]$-module

$$M(\eta) := M \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(\eta)$$

where $\mathbb{Q}_p(\eta)$ denotes the Γ_K-representation on $\mathbb{Q}_p$ given by η.

Example 1.1.6. Given a $\mathbb{Q}_p[\Gamma_K]$-module M, we have an identification $M(n) \cong M(\chi^n)$ for every $n \in \mathbb{Z}$ by Lemma 3.1.3 in Chapter II.

We assume the following generalization of Theorem 3.1.11 in Chapter II.

THEOREM 1.1.7 (Tate [**Tat67**], Sen [**Sen80**]). Let $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ be a continuous character.

(1) If $\eta(I_K)$ is finite, both $H^0(\Gamma_K, \mathbb{C}_K(\eta))$ and $H^1(\Gamma_K, \mathbb{C}_K(\eta))$ are 1-dimensional vector spaces over K.
(2) If $\eta(I_K)$ is infinite, both $H^0(\Gamma_K, \mathbb{C}_K(\eta))$ and $H^1(\Gamma_K, \mathbb{C}_K(\eta))$ vanish.

Remark. By Theorem 1.1.7, the p-adic Γ_K-representation $\mathbb{Q}_p(\eta)$ is $\mathbb{C}_K$-admissible if and only if $\eta(I_K)$ is finite. In fact, by a deep result of Sen [**Sen80**], a p-adic Γ_K-representation V is $\mathbb{C}_K$-admissible if and only if the I_K-action on V factors through a finite quotient.

LEMMA 1.1.8. The group $\chi(I_K)$ is infinite.

PROOF. We have $\ker(\chi) = \bigcap\limits_{v \geq 1} \mathrm{Gal}(K(\mu_{p^v}(\overline{K}))/K)$ as χ encodes the action of Γ_K on $\mathbb{Z}_p(1) = \varprojlim \mu_{p^v}(\overline{K})$. Let us write e_v for the ramification degree of $K(\mu_{p^v}(\overline{K}))$ over K and e for the ramification degree of K over $\mathbb{Q}_p$. We find $e_v e \geq p^{v-1}(p-1)$ by noting that $e_v e$ and $p^{v-1}(p-1)$ are respectively equal to the ramification degrees of $K(\mu_{p^v}(\overline{K}))$ and $\mathbb{Q}_p(\mu_{p^v}(\overline{K}))$ over $\mathbb{Q}_p$. We deduce that e_v grows arbitrarily large and thus obtain the desired assertion. □

Remark. Since we have $\mathbb{C}_K(n) \cong \mathbb{C}_K(\chi^n)$ for each $n \in \mathbb{Z}$ as noted in Example 1.1.6, we can deduce Theorem 3.1.11 in Chapter II from Lemma 1.1.8 and Theorem 1.1.7.

Definition 1.1.9. The *Hodge-Tate period ring* is $B_{\mathrm{HT}} := \bigoplus\limits_{n \in \mathbb{Z}} \mathbb{C}_K(n)$.

PROPOSITION 1.1.10. The Hodge-Tate period ring B_{HT} is $(\mathbb{Q}_p, \Gamma_K)$-regular.

PROOF. Let us first prove the identity $B_{\mathrm{HT}}^{\Gamma_K} = C_{\mathrm{HT}}^{\Gamma_K}$, where C_{HT} denotes the fraction field of the integral domain B_{HT}. We consider the natural action of Γ_K on $\mathbb{C}_K((t))$ with $\gamma(t) = \chi(\gamma)t$ for every $\gamma \in \Gamma_K$. Lemma 3.1.3 in Chapter II yields Γ_K-equivariant isomorphisms

$$B_{\mathrm{HT}} \simeq \mathbb{C}_K[t, t^{-1}] \quad \text{and} \quad C_{\mathrm{HT}} \simeq \mathbb{C}_K(t).$$

Since we have $B_{\mathrm{HT}}^{\Gamma_K} = K$ by Theorem 3.1.11 in Chapter II, it suffices to establish the identity $\mathbb{C}_K((t))^{\Gamma_K} = K$. The group Γ_K acts on each $f(t) = \sum c_n t^n \in \mathbb{C}_K((t))$ via the relation

$$\gamma\left(\sum c_n t^n\right) = \sum \gamma(c_n)\chi(\gamma)^n t^n \quad \text{for every } \gamma \in \Gamma_K.$$

Hence $f(t) = \sum c_n t^n \in \mathbb{C}_K((t))$ is Γ_K-invariant if and only if we have $c_n = \gamma(c_n)\chi(\gamma)^n$ for each $n \in \mathbb{Z}$ and $\gamma \in \Gamma_K$, or equivalently $c_n \in \mathbb{C}_K(n)^{\Gamma_K}$ for every $n \in \mathbb{Z}$ by Lemma 3.1.3 in Chapter II. The desired identity $\mathbb{C}_K((t))^{\Gamma_K} = K$ follows from Theorem 3.1.11 in Chapter II.

It remains to show that every nonzero $b \in B_{\mathrm{HT}}$ with $\mathbb{Q}_p b$ being stable under the Γ_K-action is a unit. Let us identify b with $f(t) = \sum c_n t^n \in \mathbb{C}_K[t, t^{-1}]$ via the Γ_K-equivariant isomorphism $B_{\mathrm{HT}} \simeq \mathbb{C}_K[t, t^{-1}]$. The group Γ_K acts continuously on B_{HT} as it acts continuously on each $\mathbb{C}_K(n)$; in particular, it acts on $f(t)$ via a continuous character $\eta : \Gamma_K \to \mathbb{Q}_p^\times$. For each $n \in \mathbb{Z}$ and $\gamma \in \Gamma_K$, we find $\eta(\gamma)c_n = \gamma(c_n)\chi(\gamma)^n$ or equivalently $c_n = (\eta^{-1}\chi^n)(\gamma)\gamma(c_n)$. Hence we have $c_n \in \mathbb{C}_K(\eta^{-1}\chi^n)^{\Gamma_K}$ for every $n \in \mathbb{Z}$ and in turn deduce from Theorem 1.1.7 that $(\eta^{-1}\chi^n)(I_K)$ is finite for every $n \in \mathbb{Z}$ with $c_n \neq 0$. Let us now choose $m \in \mathbb{Z}$ with $c_m \neq 0$. If we have $c_n \neq 0$ for some $n \neq m$, we see that the image of I_K under $\chi^{n-m} = (\eta^{-1}\chi^n) \cdot (\eta^{-1}\chi^m)^{-1}$ must be finite, which contradicts Lemma 1.1.8. We find that $f(t) = c_m t^m \in \mathbb{C}_K[t, t^{-1}]$ is a unit, thereby completing the proof. □

PROPOSITION 1.1.11. A p-adic representation V of Γ_K is Hodge-Tate if and only if it is B_{HT}-admissible.

PROOF. Since we have

$$D_{B_{\mathrm{HT}}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{\Gamma_K} = \bigoplus_{n \in \mathbb{Z}} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K}, \tag{1.1}$$

the desired assertion follows from Proposition 3.1.12 in Chapter II. □

Example 1.1.12. Given a p-adic Γ_K-representation V which fits into an exact sequence

$$0 \longrightarrow \mathbb{Q}_p(m) \longrightarrow V \longrightarrow \mathbb{Q}_p(n) \longrightarrow 0$$

with $m \neq n$, we assert that V is Hodge-Tate. For every $i \in \mathbb{Z}$, we have an exact sequence

$$0 \longrightarrow \mathbb{C}_K(i+m) \longrightarrow V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(i) \longrightarrow \mathbb{C}_K(i+n) \longrightarrow 0$$

which gives rise to a long exact sequence

$$0 \longrightarrow \mathbb{C}_K(i+m)^{\Gamma_K} \longrightarrow (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(i))^{\Gamma_K} \longrightarrow \mathbb{C}_K(i+n)^{\Gamma_K} \longrightarrow H^1(\Gamma_K, \mathbb{C}_K(i+m)).$$

Therefore Theorem 3.1.11 in Chapter II yields an identification

$$(V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(i))^{\Gamma_K} \cong \begin{cases} K & \text{for } i = -m, -n, \\ 0 & \text{for } i \neq -m, -n. \end{cases}$$

Now we find

$$\dim_K D_{B_{\mathrm{HT}}}(V) = \sum_{i \in \mathbb{Z}} \dim_K (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(i))^{\Gamma_K} = 2 = \dim_{\mathbb{Q}_p} V$$

and in turn establish the desired assertion.

Remark. On the other hand, a self extension of $\mathbb{Q}_p$ is not necessarily Hodge-Tate. For example, we can show that the two-dimensional $\mathbb{Q}_p$-vector space with the Γ_K-action given by the matrix $\begin{pmatrix} 1 & \log_p \circ \chi \\ 0 & 1 \end{pmatrix}$ is not Hodge-Tate, where $\log_p$ denotes the Iwasawa logarithm.

PROPOSITION 1.1.13. For a continuous character $\eta : \Gamma_K \to \mathbb{Q}_p^\times$, the Γ_K-representation $\mathbb{Q}_p(\eta)$ is Hodge-Tate if and only if there exists some $n \in \mathbb{Z}$ with $(\eta\chi^n)(I_K)$ finite.

PROOF. By Proposition 3.1.12 in Chapter II, the Γ_K-representation $\mathbb{Q}_p(\eta)$ is Hodge-Tate if and only if there exists an integer n with $(\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} \neq 0$, or equivalently $\mathbb{C}_K(\eta\chi^n)^{\Gamma_K} \neq 0$ by Example 1.1.6. Hence the assertion follows from Theorem 1.1.7. □

Definition 1.1.14. Given a Hodge-Tate Γ_K-representation V, an integer n is a *Hodge-Tate weight* of V with multiplicity m if we have

$$\dim_K (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K} = m > 0.$$

Remark. Readers should be aware that many authors use the opposite sign convention for Hodge-Tate weights. We will explain the reason for our choice in Proposition 2.4.4.

Example 1.1.15. We record the Hodge-Tate weights for some Hodge-Tate representations.

(1) For every $n \in \mathbb{Z}$, the Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ has Hodge-Tate weight $-n$.
(2) For a p-divisible group G over $\mathcal{O}_K$, the rational Tate module $V_p(G)$ has Hodge-Tate weights 0 or -1 (possibly both) by Theorem 3.3.13 in Chapter II.
(3) For an abelian variety A over K with good reduction, the étale cohomology group $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ has Hodge-Tate weights $0, 1, \cdots, n$ by Theorem 3.3.17 in Chapter II.

1.2. Formal properties of admissible representations

Throughout this subsection, we fix a $(\mathbb{Q}_p, \Gamma_K)$-regular ring B and write $E := B^{\Gamma_K}$. In addition, we denote by $\mathrm{Rep}_{\mathbb{Q}_p}^{B}(\Gamma_K)$ the category of B-admissible Γ_K-representations.

THEOREM 1.2.1. Let V be a p-adic Γ_K-representation.

(1) There exists a natural map
$$\alpha_V : D_B(V) \otimes_E B \longrightarrow V \otimes_{\mathbb{Q}_p} B$$
which is B-linear, Γ_K-equivariant, and injective.

(2) V satisfies the inequality
$$\dim_E D_B(V) \leq \dim_{\mathbb{Q}_p} V \tag{1.2}$$
with equality precisely when α_V is an isomorphism.

PROOF. Let us first consider statement (1). We have the natural map
$$\alpha_V : D_B(V) \otimes_E B \longrightarrow (V \otimes_{\mathbb{Q}_p} B) \otimes_E B \cong V \otimes_{\mathbb{Q}_p} (B \otimes_E B) \longrightarrow V \otimes_{\mathbb{Q}_p} B,$$
which is clearly B-linear and Γ_K-equivariant. We wish to show that α_V is injective. Since the fraction field C of B is $(\mathbb{Q}_p, \Gamma_K)$-regular as noted in Example 1.1.2, we obtain a canonical C-linear map
$$\beta_V : D_C(V) \otimes_E C \longrightarrow V \otimes_{\mathbb{Q}_p} C$$
which fits into a commutative diagram
$$\begin{array}{ccc} D_B(V) \otimes_E B & \xrightarrow{\alpha_V} & V \otimes_{\mathbb{Q}_p} B \\ \big\downarrow & & \big\downarrow \\ D_C(V) \otimes_E C & \xrightarrow{\beta_V} & V \otimes_{\mathbb{Q}_p} C \end{array}$$
with injective vertical maps. It suffices to prove that β_V is injective. Suppose for contradiction that $\ker(\beta_V)$ is nonzero. Take an E-basis (e_i) of $D_C(V) = (V \otimes_{\mathbb{Q}_p} C)^{\Gamma_K}$ and choose a nontrivial C-linear relation $\sum c_i e_i = 0$ with minimal number of nonzero terms. We may set $c_j = 1$ for some j. For every $\gamma \in \Gamma_K$, we find
$$\sum (\gamma(c_i) - c_i) e_i = \gamma\left(\sum c_i e_i\right) - \sum c_i e_i = 0 \quad \text{and} \quad \gamma(c_j) - c_j = \gamma(1) - 1 = 0.$$
By the minimality of our relation, each c_i satisfies the equality $c_i = \gamma(c_i)$ for every $\gamma \in \Gamma_K$ and thus lies in $C^{\Gamma_K} = E$. Now we have a nontrivial E-linear relation $\sum c_i e_i = 0$ for the E-basis (e_i) of $D_C(V)$, thereby obtaining a desired contradiction.

It remains to verify statement (2). Since the inequality (1.2) is evident by statement (1), we only need to consider the equality condition. If α_V is an isomorphism, the inequality becomes an equality. For the converse, we henceforth assume the identity $\dim_E D_B(V) = \dim_{\mathbb{Q}_p} V$. Let us choose an E-basis (u_i) of $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ and a $\mathbb{Q}_p$-basis (v_i) of V. We may represent α_V by a $d \times d$ matrix M_V with $d = \dim_E D_B(V) = \dim_{\mathbb{Q}_p} V$. We wish to show that $\det(M_V)$ is a unit in B. We have $\det(M_V) \neq 0$ as the map $D_B(V) \otimes_E C \to V \otimes_{\mathbb{Q}_p} C$ induced by α_V is an isomorphism for being an injective map between vector spaces of equal dimension. Meanwhile, Γ_K acts trivially on $u_1 \wedge \cdots \wedge u_d$ and via some $\mathbb{Q}_p$-valued character η on $v_1 \wedge \cdots \wedge v_d$. Since the Γ_K-equivariant map α_V yields the identity
$$(\wedge^d \alpha_V)(u_1 \wedge \cdots \wedge u_d) = \det(M_V)(v_1 \wedge \cdots \wedge v_d),$$
we deduce that Γ_K acts on $\det(M_V)$ via η^{-1} and in turn find $\det(M_V) \in B^\times$ as desired. $\square$

PROPOSITION 1.2.2. The functor D_B is exact and faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$.

PROOF. Let V and W be arbitrary B-admissible Γ_K-representations. Theorem 1.2.1 yields natural Γ_K-equivariant B-linear isomorphisms

$$D_B(V) \otimes_E B \cong V \otimes_{\mathbb{Q}_p} B \quad \text{and} \quad D_B(W) \otimes_E B \cong W \otimes_{\mathbb{Q}_p} B.$$

Given $f \in \mathrm{Hom}_{\mathbb{Q}_p[\Gamma_K]}(V, W)$ with the associated map $D_B(f) : D_B(V) \to D_B(W)$ being zero, we observe that the map $V \otimes_{\mathbb{Q}_p} B \to W \otimes_{\mathbb{Q}_p} B$ induced by f is zero and in turn deduce that f must be zero. Therefore the functor D_B is faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$.

It remains to verify that D_B is exact on $\mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$. Let us consider an arbitrary short exact sequence of B-admissible Γ_K-representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$

We obtain a short exact sequence

$$0 \longrightarrow U \otimes_{\mathbb{Q}_p} B \longrightarrow V \otimes_{\mathbb{Q}_p} B \longrightarrow W \otimes_{\mathbb{Q}_p} B \longrightarrow 0,$$

which we naturally identify with a short exact sequence

$$0 \longrightarrow D_B(U) \otimes_E B \longrightarrow D_B(V) \otimes_E B \longrightarrow D_B(W) \otimes_E B \longrightarrow 0$$

by Theorem 1.2.1. The desired assertion is now evident as B is faithfully flat over the field E by a standard fact stated in the Stacks project [**Sta**, Tag 00HQ]. □

Remark. The functor D_B is not fully faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$ with values in the category of vector spaces over E; indeed, the isomorphism class of $D_B(V)$ for every $V \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$ depends only on the dimension of V. In practice, however, we enhance D_B to a functor that takes values in a category of vector spaces over E with some additional structures, as briefly described in Chapter I, §2.1. We will see in §3 that such an enhanced functor is fully faithful for the crystaline period ring $B = B_{\mathrm{cris}}$.

PROPOSITION 1.2.3. The category $\mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$ is closed under taking subquotients.

PROOF. Consider a short exact sequence of p-adic Γ_K-representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0 \tag{1.3}$$

with $V \in \mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$. We wish to show that both U and W are B-admissible. We note that the functor D_B is left exact on $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ by construction and thus obtain an exact sequence

$$0 \longrightarrow D_B(U) \longrightarrow D_B(V) \longrightarrow D_B(W). \tag{1.4}$$

In addition, by Theorem 1.2.1 we have

$$\dim_E D_B(U) \leq \dim_{\mathbb{Q}_p} U \quad \text{and} \quad \dim_E D_B(W) \leq \dim_{\mathbb{Q}_p} W.$$

Now the exact sequences (1.3) and (1.4) together yield the relation

$$\dim_E D_B(V) \leq \dim_E D_B(U) + \dim_E D_B(W) \leq \dim_{\mathbb{Q}_p} U + \dim_{\mathbb{Q}_p} W = \dim_{\mathbb{Q}_p} V.$$

Since V is B-admissible, all inequalities are in fact equalities. Therefore we deduce that both U and W are B-admissible as desired. □

Remark. In general, the category $\mathrm{Rep}_{\mathbb{Q}_p}^B(\Gamma_K)$ is not closed under taking extensions. For example, the category of Hodge-Tate representations is not closed under taking extensions by the remark following Example 1.1.12.

Proposition 1.2.4. Given B-admissible Γ_K-representations V and W, their tensor product $V \otimes_{\mathbb{Q}_p} W$ is B-admissible with a natural isomorphism

$$D_B(V) \otimes_E D_B(W) \cong D_B(V \otimes_{\mathbb{Q}_p} W).$$

Proof. Theorem 1.2.1 yields natural Γ_K-equivariant B-linear isomorphisms

$$\alpha_V : D_B(V) \otimes_E B \xrightarrow{\sim} V \otimes_{\mathbb{Q}_p} B \quad \text{and} \quad \alpha_W : D_B(W) \otimes_E B \xrightarrow{\sim} W \otimes_{\mathbb{Q}_p} B.$$

Let us consider the natural map

$$D_B(V) \otimes_E D_B(W) \longrightarrow (V \otimes_{\mathbb{Q}_p} B) \otimes_E (W \otimes_{\mathbb{Q}_p} B) \longrightarrow (V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B$$

with the first arrow given by the identifications

$$D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K} \quad \text{and} \quad D_B(W) = (W \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}.$$

Since the second arrow is evidently Γ_K-equivariant, we obtain a natural E-linear map

$$D_B(V) \otimes_E D_B(W) \longrightarrow \big((V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B\big)^{\Gamma_K} \cong D_B(V \otimes_{\mathbb{Q}_p} W). \tag{1.5}$$

It is not hard to see that this map is injective; indeed, this map extends to a B-linear map

$$(D_B(V) \otimes_E D_B(W)) \otimes_E B \longrightarrow \big((V \otimes_{\mathbb{Q}_p} B) \otimes_E (W \otimes_{\mathbb{Q}_p} B)\big) \otimes_E B \longrightarrow (V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B$$

which coincides with the isomorphism $\alpha_V \otimes \alpha_W$ under the identifications

$$(D_B(V) \otimes_E D_B(W)) \otimes_E B \cong (D_B(V) \otimes_E B) \otimes_B (D_B(W) \otimes_E B),$$
$$\big((V \otimes_{\mathbb{Q}_p} B) \otimes_E (W \otimes_{\mathbb{Q}_p} B)\big) \otimes_E B \cong (V \otimes_{\mathbb{Q}_p} B \otimes_E B) \otimes_B (W \otimes_{\mathbb{Q}_p} B \otimes_E B),$$
$$(V \otimes_{\mathbb{Q}_p} W) \otimes_{\mathbb{Q}_p} B \cong (V \otimes_{\mathbb{Q}_p} B) \otimes_B (W \otimes_{\mathbb{Q}_p} B).$$

Therefore we obtain the inequality

$$\dim_E D_B(V \otimes_{\mathbb{Q}_p} W) \geq (\dim_E D_B(V)) \cdot (\dim_E D_B(W)) = \dim_{\mathbb{Q}_p} V \otimes_{\mathbb{Q}_p} W$$

where the equality follows from the B-admissibility of V and W. We find by Theorem 1.2.1 that this inequality is indeed an equality and in turn deduce that $V \otimes_{\mathbb{Q}_p} W$ is B-admissible with the natural isomorphism (1.5). □

Proposition 1.2.5. Given a B-admissible Γ_K-representation V and a positive integer n, both $\wedge^n(V)$ and $\mathrm{Sym}^n(V)$ are B-admissible with natural filtered isomorphisms

$$\wedge^n(D_B(V)) \cong D_B(\wedge^n(V)) \quad \text{and} \quad \mathrm{Sym}^n(D_B(V)) \cong D_B(\mathrm{Sym}^n(V)).$$

Proof. Let us only consider exterior powers here, as the same argument works with symmetric powers. Proposition 1.2.4 implies that $V^{\otimes n}$ is B-admissible with a natural isomorphism $D_B(V^{\otimes n}) \cong D_B(V)^{\otimes n}$. We find that $\wedge^n(V)$ is B-admissible by Proposition 1.2.3 and in turn obtain a natural surjective E-linear map

$$D_B(V)^{\otimes n} \xrightarrow{\sim} D_B(V^{\otimes n}) \twoheadrightarrow D_B(\wedge^n(V))$$

by Proposition 1.2.2. It is straightforward to check that this map factors through the natural surjection $D_B(V)^{\otimes n} \twoheadrightarrow \wedge^n(D_B(V))$. Hence we have a natural surjective E-linear map

$$\wedge^n(D_B(V)) \twoheadrightarrow D_B(\wedge^n(V)),$$

which turns out to be an isomorphism since we have

$$\dim_E \wedge^n(D_B(V)) = \dim_E D_B(\wedge^n(V))$$

by the B-admissibility of V and $\wedge^n(V)$. □

LEMMA 1.2.6. Every d-dimensional p-adic Γ_K-representation V admits natural Γ_K-equivariant $\mathbb{Q}_p$-linear isomorphisms

$$\Delta : \det(V^\vee) \xrightarrow{\sim} \det(V)^\vee \quad \text{ and } \quad \Lambda : \det(V^\vee) \otimes_{\mathbb{Q}_p} \wedge^{d-1} V \xrightarrow{\sim} V^\vee.$$

PROOF. Take arbitrary elements $f_1, \cdots, f_d \in V^\vee$ and $v_1, \cdots, v_d \in V$. Let M denote the $d \times d$ matrix whose (i, j)-entry is $f_i(v_j)$. We obtain Δ and Λ as $\mathbb{Q}_p$-linear maps with

$$\Delta(f_1 \wedge \cdots \wedge f_d)(v_1 \wedge \cdots \wedge v_d) = \det(M),$$
$$\Lambda\left((f_1 \wedge \cdots \wedge f_d) \otimes (v_2 \wedge \cdots \wedge v_d)\right)(v_1) = \det(M).$$

It is straightforward to verify that Δ and Λ are Γ_K-equivariant isomorphisms. ☐

PROPOSITION 1.2.7. For every B-admissible Γ_K-representation V, the dual representation $V^\vee$ is B-admissible with a natural E-linear perfect pairing

$$D_B(V) \otimes_E D_B(V^\vee) \cong D_B(V \otimes_{\mathbb{Q}_p} V^\vee) \longrightarrow D_B(\mathbb{Q}_p) \cong E. \tag{1.6}$$

PROOF. Let us first consider the case where V has dimension 1 over $\mathbb{Q}_p$. We fix a nonzero vector $v \in V$ and take $f \in V^\vee = \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$ with $f(v) = 1$. In addition, we represent the Γ_K-action on V by a continuous character $\eta : \Gamma_K \to \mathbb{Q}_p^\times$. We obtain the equalities

$$\gamma(v) = \eta(\gamma)v \quad \text{ and } \quad \gamma(f) = \eta(\gamma)^{-1} f \quad \text{ for every } \gamma \in \Gamma_K.$$

Since $D_B(V) = (V \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ is 1-dimensional over E by the B-admissibility of V, it admits a basis given by a vector $v \otimes b$ for some $b \in B$. Now we find

$$v \otimes b = \gamma(v \otimes b) = \gamma(v) \otimes \gamma(b) = \eta(\gamma)v \otimes \gamma(b) = v \otimes \eta(\gamma)\gamma(b) \quad \text{ for every } \gamma \in \Gamma_K$$

or equivalently

$$b = \eta(\gamma)\gamma(b) \quad \text{ for every } \gamma \in \Gamma_K.$$

Moreover, we have $b \in B^\times$ as $v \otimes b$ yields a B-basis for $V \otimes_{\mathbb{Q}_p} B$ via the natural isomorphism $D_B(V) \otimes_E B \cong V \otimes_{\mathbb{Q}_p} B$ given by Theorem 1.2.1. Hence $D_B(V^\vee) = (V^\vee \otimes_{\mathbb{Q}_p} B)^{\Gamma_K}$ contains a nonzero vector $f \otimes b^{-1}$. We deduce that the inequality

$$\dim_E D_B(V^\vee) \leq \dim_{\mathbb{Q}_p} V^\vee = 1$$

given by Theorem 1.2.1 must be an equality, which means that $V^\vee$ is B-admissible. We also observe that $f \otimes b^{-1}$ yields an E-basis for $D_B(V^\vee)$ and in turn find that the map (1.6) is a perfect pairing.

We now consider the general case. Let us write $d := \dim_{\mathbb{Q}_p} V$ for notational convenience. Proposition 1.2.5 implies that both $\det(V) = \wedge^d V$ and $\wedge^{d-1} V$ are B-admissible. Since $\det(V)$ has dimension 1 over $\mathbb{Q}_p$, we deduce from Proposition 1.2.4 and Lemma 1.2.6 that $V^\vee$ is B-admissible. Hence it remains to prove that the map (1.6) is a perfect pairing. We have

$$d = \dim_E D_B(V) = \dim_E D_B(V^\vee)$$

by the B-admissibility of V and $V^\vee$. Upon choosing E-bases for $D_B(V)$ and $D_B(V^\vee)$, we can represent the pairing (1.6) by a $d \times d$ matrix M. It suffices to show that $\det(M)$ is nonzero, or equivalently that the induced pairing

$$\det(D_B(V)) \otimes_E \det(D_B(V^\vee)) \longrightarrow E$$

is perfect. Proposition 1.2.5 yields natural isomorphisms

$$\det(D_B(V)) \cong D_B(\det(V)) \quad \text{ and } \quad \det(D_B(V^\vee)) \cong D_B(\det(V^\vee)).$$

Hence the desired assertion is evident by our discussion in the first paragraph. ☐

2. de Rham representations

In this section, we define and study the de Rham period ring and de Rham representations. The primary references for this section are the notes of Brinon-Conrad [**BC**, §4 and §6] and the article of Scholze [**Sch12**].

2.1. Perfectoid fields and their tilts

Let us begin with the notion of perfectoid fields, which provides a modern perspective of Fontaine's original work.

Definition 2.1.1. A *perfectoid field* is a complete nonarchimedean field C of residue characteristic p with the following properties:

(i) The valuation on C is nondiscrete.
(ii) The p-th power map on $\mathcal{O}_C/p\mathcal{O}_C$ is surjective.

Remark. By convention, we assume that the valuation on a nonarchimedean field is not trivial. On the other hand, the valuation on a valued field may be trivial.

LEMMA 2.1.2. Let C be a complete nonarchimedean field of residue characteristic p. If the p-th power map on C is surjective, the field C is a perfectoid field.

PROOF. Let us denote by ν the valuation on C and take an arbitrary element $x \in C$. Since the p-th power map on C is surjective by our assumption, there exists an element $y \in C$ with $x = y^p$. If x has positive valuation, we find

$$0 < \nu(y) = \nu(x)/p < \nu(x). \tag{2.1}$$

We deduce that C does not have an element with minimum positive valuation, which in particular implies that the valuation ν is not discrete. Moreover, we note that the p-th power map on $\mathcal{O}_C$ is surjective; indeed, if x lies in $\mathcal{O}_C$ we have $y \in \mathcal{O}_C$ by the relation (2.1). Hence the p-th power map on $\mathcal{O}_C/p\mathcal{O}_C$ is also surjective. The desired assertion is now evident. □

Remark. The converse of Lemma 2.1.2 does not hold; in other words, the p-th power map on a perfectoid field is not neccessarily surjective.

Example 2.1.3. Since $\mathbb{C}_K$ is algebraically closed as noted in Chapter II, Proposition 3.1.10, it is a perfectoid field by Lemma 2.1.2.

Remark. In fact, Lemma 2.1.2 shows that every complete nonarchimedean algebraically closed field of residue characteristic p is a perfectoid field.

PROPOSITION 2.1.4. A nonarchimedean field of characteristic p is perfectoid if and only if it is complete and perfect.

PROOF. By definition, every perfectoid field of characteristic p is complete and perfect. Conversely, every complete nonarchimedean perfect field of characteristic p is perfectoid by Lemma 2.1.2. □

Definition 2.1.5. Let C be a perfectoid field.

(1) The *tilt* of C is $C^\flat := \varprojlim_{x \mapsto x^p} C$ endowed with the natural multiplication.
(2) The *sharp map associated to* C is the map $C^\flat \to C$ which sends each $c = (c_n) \in C^\flat$ to the first component $c^\sharp = c_0$.

For the rest of this subsection, we fix a perfectoid field C with a valuation ν. We aim to show that the multiplicative monoid $C^\flat$ is naturally a perfectoid field of characteristic p.

PROPOSITION 2.1.6. Fix an element $\varpi \in C^\times$ with $0 < \nu(\varpi) \leq \nu(p)$.

(1) Given arbitrary elements $x, y \in \mathcal{O}_C$ with $x - y \in \varpi\mathcal{O}_C$ we have
$$x^{p^n} - y^{p^n} \in \varpi^{n+1}\mathcal{O}_C \quad \text{for each integer } n \geq 0.$$
(2) The natural projection $\mathcal{O}_C \twoheadrightarrow \mathcal{O}_C/\varpi\mathcal{O}_C$ induces a multiplicative bijection
$$\varprojlim_{x\mapsto x^p} \mathcal{O}_C \cong \varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C. \tag{2.2}$$
(3) The monoid $\varprojlim_{x\mapsto x^p} \mathcal{O}_C$ is naturally a ring of characteristic p via the map (2.2).

PROOF. The inequality $\nu(\varpi) \leq \nu(p)$ implies that p is divisible by ϖ in $\mathcal{O}_C$. In addition, for elements $x, y \in \mathcal{O}_C$ and an integer $n \geq 1$ we find
$$x^{p^n} - y^{p^n} = \left(y^{p^{n-1}} + (x^{p^{n-1}} - y^{p^{n-1}})\right)^p - y^{p^n}.$$
Hence we obtain statement (1) by a simple induction.

Let us now consider statement (2). We wish to construct an inverse map
$$f: \varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C \longrightarrow \varprojlim_{x\mapsto x^p} \mathcal{O}_C.$$
Take an arbitrary element $\overline{c} = (\overline{c}_n) \in \varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$ and choose a lift $c_n \in \mathcal{O}_C$ of each $\overline{c}_n$. We have
$$c_{n+m+l}^{p^l} - c_{n+m} \in \varpi\mathcal{O}_C \quad \text{for all } l, m, n \geq 0$$
and consequently find
$$c_{n+m+l}^{p^{m+l}} - c_{n+m}^{p^m} \in \varpi^{m+1}\mathcal{O}_C \quad \text{for all } n, m \geq 0$$
by statement (1). We see that for each $n \geq 0$ the sequence $(c_{n+m}^{p^m})_{m\geq 0}$ converges in $\mathcal{O}_C$ for being Cauchy. In addition, statement (1) implies that the limit of the sequence $(c_{n+m}^{p^m})_{m\geq 0}$ for each $n \geq 0$ does not depend on the choice of c_n. Now we write
$$f_n(\overline{c}) := \lim_{m\to\infty} c_{n+m}^{p^m} \quad \text{for each } n \geq 0$$
and obtain the desired inverse by setting
$$f(\overline{c}) := (f_n(\overline{c})) \in \varprojlim_{x\mapsto x^p} \mathcal{O}_C.$$

It remains to verify statement (3). Since ϖ divides p in $\mathcal{O}_C$ as already noted in the first paragraph, the ring $\mathcal{O}_C/\varpi\mathcal{O}_C$ is of characteristic p and thus induces a natural ring structure on $\varprojlim_{x\mapsto x^p} \mathcal{O}_C \cong \varprojlim_{x\mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$. Moreover, this ring structure does not depend on ϖ; indeed, for arbitrary elements $a = (a_n)$ and $b = (b_n)$ in $\varprojlim_{x\mapsto x^p} \mathcal{O}_C$ we find
$$ab = (a_n b_n) \quad \text{and} \quad a + b = \left(\lim_{m\to\infty} (a_{m+n} + b_{m+n})^{p^m}\right).$$
Now we establish statement (3) as $\varprojlim_{x\mapsto x^p} \mathcal{O}_C$ is evidently of characteristic p. □

Remark. If C has characteristic 0, it is customary to take $\varpi = p$.

PROPOSITION 2.1.7. The tilt $C^\flat$ of C is naturally a field of characteristic p which satisfies the following properties:

(i) It is complete under the natural valuation $\nu^\flat$ with $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in C^\flat$.

(ii) For every $\varpi \in C^\times$ with $0 < \nu(\varpi) \leq \nu(p)$, there exists a natural identification

$$\mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C.$$

PROOF. Let us fix an element $\varpi \in C^\times$ with $0 < \nu(\varpi) \leq \nu(p)$. Proposition 2.1.6 shows that $\mathcal{O} := \varprojlim_{x \mapsto x^p} \mathcal{O}_C$ is naturally a ring of characteristic p with a canonical identification

$$\mathcal{O} \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C. \tag{2.3}$$

We may identify $C^\flat$ with the fraction field of $\mathcal{O}$, which is evidently perfect of characteristic p.

We assert that the function $\nu^\flat$ on $C^\flat$ with $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in C^\flat$ is indeed a valuation. It is clear by construction that $\nu^\flat$ is a monoid homomorphism with respect to the multiplication on $C^\flat$. Let us take arbitrary elements $a = (a_n)$ and $b = (b_n)$ in $C^\flat$. Without loss of generality, we may assume the inequality $\nu^\flat(a) \geq \nu^\flat(b)$. Since we have

$$\nu(a_n) = \frac{1}{p^n}\nu(a_0) = \frac{1}{p^n}\nu^\flat(a) \geq \frac{1}{p^n}\nu^\flat(b) = \frac{1}{p^n}\nu(b_0) = \nu(b_n) \quad \text{for each } n \geq 0,$$

we write $a = bu$ for some $u \in \mathcal{O}$ and find

$$\nu^\flat(a+b) = \nu^\flat((u+1)b) = \nu^\flat(u+1) + \nu^\flat(b) \geq \nu^\flat(b) = \min(\nu^\flat(a), \nu^\flat(b))$$

where the inequality follows from the observation that $u+1$ is an element of $\mathcal{O}$. Therefore we deduce that $\nu^\flat$ is a valuation as desired.

Let us now take an arbitrary element $c = (c_n) \in C^\flat$. We find

$$\nu(c_n) = \frac{1}{p^n}\nu(c_0) = \frac{1}{p^n}\nu^\flat(c) \quad \text{for each } n \geq 0$$

and in turn verify that $\mathcal{O}$ is the valuation ring of $C^\flat$. Moreover, given an integer $m > 0$ we have $\nu(c_n) \geq \nu(\varpi)$ for each $n \leq m$ if and only if c satisfies the inequality $\nu^\flat(c) \geq p^m\nu(\varpi)$. Hence the map (2.3) is a topological isomorphism with respect to the $\nu^\flat$-adic topology on $\mathcal{O}$ and the inverse limit topology on $\varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$. It is not hard to see that $\varprojlim_{x \mapsto x^p} \mathcal{O}_C/\varpi\mathcal{O}_C$ is complete, which implies that both $\mathcal{O}_{C^\flat} = \mathcal{O}$ and $C^\flat$ are complete. □

Remark. Proposition 2.1.6 and Proposition 2.1.7 remain valid if we replace C by an arbitrary complete nonarchimedean field L with its "tilt" $L^\flat := \varprojlim_{c \mapsto c^p} L$. However, if L is not perfectoid the valuation on $L^\flat$ may be trivial. For example, if L is a p-adic field $L^\flat$ is isomorphic to its residue field with the trivial valuation.

PROPOSITION 2.1.8. The sharp map associated to C is continuous on $\mathcal{O}_{C^\flat}$.

PROOF. Proposition 2.1.7 yields a topological isormohpsim

$$\mathcal{O}_{C^\flat} \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p\mathcal{O}_C.$$

Given an integer $m \geq 1$, if we have elements $a = (a_n)$ and $b = (b_n)$ in $\mathcal{O}_{C^\flat}$ with $a_n = b_n$ for each $n \leq m$, we apply Proposition 2.1.6 to find $a^\sharp - b^\sharp \in p^{m+1}\mathcal{O}_C$. Therefore we establish the desired assertion. □

LEMMA 2.1.9. For every $c \in \mathcal{O}_C$ there exists an element $c^\flat \in \mathcal{O}_{C^\flat}$ with $c - (c^\flat)^\sharp \in p\mathcal{O}_C$.

PROOF. Proposition 2.1.7 yields a natural isormohpsim

$$\mathcal{O}_{C^\flat} \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p\mathcal{O}_C.$$

Let $\overline{c}$ denote the image of c in $\mathcal{O}_C/p\mathcal{O}_C$. Since the p-th power map on $\mathcal{O}_C/p\mathcal{O}_C$ is surjective, we obtain the desired assertion by taking $c^\flat = (c_n^\flat) \in \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p\mathcal{O}_C \cong \mathcal{O}_{C^\flat}$ with $c_0^\flat = \overline{c}$. □

PROPOSITION 2.1.10. The map $\mathcal{O}_{C^\flat} \to \mathcal{O}_C/p\mathcal{O}_C$ which sends each $c \in \mathcal{O}_{C^\flat}$ to the image of $c^\sharp$ in $\mathcal{O}_C/p\mathcal{O}_C$ is a surjective ring homomorphism.

PROOF. Since we have $\mathcal{O}_{C^\flat} = \varprojlim_{x \mapsto x^p} \mathcal{O}_C$ as noted in Proposition 2.1.7, the assertion is straightforward to verify by Proposition 2.1.6 and Lemma 2.1.9. □

Remark. The sharp map associated to C is a multiplicative map but is not a ring homomorphism unless C is of characteristic p.

PROPOSITION 2.1.11. The valued fields C and $C^\flat$ have the same value groups.

PROOF. Let $\nu^\flat$ denote the valuation on $C^\flat$. Since we have $\nu^\flat\left((C^\flat)^\times\right) \subseteq \nu(C^\times)$ by Proposition 2.1.7, we only need to establish the relation $\nu(C^\times) \subseteq \nu^\flat\left((C^\flat)^\times\right)$. Consider an arbitrary element $c \in C^\times$. We wish to find an element $c^\flat \in (C^\flat)^\times$ with $\nu^\flat(c^\flat) = \nu(c)$. As we know that ν is nondiscrete, we can choose an element $\varpi \in \mathcal{O}_C$ with $0 < \nu(\varpi) < \nu(p)$. Let us write $c = \varpi^n u$ for some $n \in \mathbb{Z}$ and $u \in \mathcal{O}_C$ with $\nu(u) < \nu(\varpi)$. Lemma 2.1.9 yields elements $\varpi^\flat, u^\flat \in \mathcal{O}_{C^\flat}$ with $\varpi - (\varpi^\flat)^\sharp \in p\mathcal{O}_C$ and $u - (u^\flat)^\sharp \in p\mathcal{O}_C$. By Proposition 2.1.7, we find

$$\begin{aligned}
\nu^\flat(\varpi^\flat) &= \nu((\varpi^\flat)^\sharp) = \nu\left(\varpi - (\varpi - (\varpi^\flat)^\sharp)\right) = \nu(\varpi),\\
\nu^\flat(u^\flat) &= \nu((u^\flat)^\sharp) = \nu\left(u - (u - (u^\flat)^\sharp)\right) = \nu(u).
\end{aligned}$$

Hence we take $c^\flat = (\varpi^\flat)^n u^\flat$ and obtain the equality $\nu^\flat(c^\flat) = \nu(c)$ as desired. □

PROPOSITION 2.1.12. The field $C^\flat$ is a perfectoid field of characteristic p.

PROOF. Proposition 2.1.11 implies that the value group of $C^\flat$ is not trivial. Since $C^\flat$ is perfect by construction, the assertion follows from Proposition 2.1.4 and Proposition 2.1.7. □

Remark. A main result of Scholze [**Sch12**] establishes a canonical bijection between the finite extensions of C and the finite extensions of $C^\flat$, called the *tilting equivalence*. In Chapter V, we will exploit this equivalence to present a classification of all p-adic Γ_K-representations in terms of certain modules over a field of characteristic p.

Example 2.1.13. The field $\mathbb{C}_K$ is perfectoid as noted in Example 2.1.3 and thus gives rise to a perfectoid field $F := \mathbb{C}_K^\flat$ of characteristic p by Proposition 2.1.12.

Remark. Since $\mathbb{C}_K$ is algebraically closed by Proposition 3.1.10 in Chapter II, the tilting equivalence implies that F is algebraically closed. We will prove this fact in Chapter IV.

PROPOSITION 2.1.14. If C is of characteristic p, there exists a natural identification $C^\flat \cong C$.

PROOF. The assertion is evident as C is perfect by Proposition 2.1.4. □

2.2. The de Rham period ring B_{dR}

For the rest of this chapter, we denote by ν the normalized p-adic valuation on $\mathbb{C}_K$ and by $\nu^\flat$ the valuation on $F = \mathbb{C}_K^\flat$ with $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in F$.

LEMMA 2.2.1. The ring $\mathcal{O}_F$ is a perfect $\mathbb{F}_p$-algebra.

PROOF. The assertion is evident by Proposition 2.1.4 and Proposition 2.1.12. ☐

Definition 2.2.2. The *infinitesimal period ring* is $A_{\mathrm{inf}} := W(\mathcal{O}_F)$.

Remark. Our definition of A_{inf} relies on Lemma 2.2.1.

LEMMA 2.2.3. The ring A_{inf} is an integral domain.

PROOF. Since A_{inf} is naturally a subring of $W(F)$, we deduce the desired assertion from Lemma 2.3.9 in Chapter II. ☐

PROPOSITION 2.2.4. There exists a surjective ring homomorphism $\theta : A_{\mathrm{inf}} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}$ with

$$\theta\left(\sum_{n=0}^{\infty}[c_n]p^n\right) = \sum_{n=0}^{\infty} c_n^\sharp p^n \quad \text{for all } c_n \in \mathcal{O}_F. \tag{2.4}$$

PROOF. Proposition 2.1.10 yields a surjective ring homomorphism $\overline{\theta} : \mathcal{O}_F \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ with $\overline{\theta}(c) = \overline{c^\sharp}$ for each $c \in \mathcal{O}_F$, where $\overline{c^\sharp}$ denotes the image of $c^\sharp$ in $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. Moreover, by construction $\overline{\theta}$ lifts to a multiplicative map $\hat{\theta} : \mathcal{O}_F \to \mathcal{O}_{\mathbb{C}_K}$ with $\hat{\theta}(c) = c^\sharp$ for each $c \in \mathcal{O}_F$. Hence we obtain a ring homomorphism $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_K}$ which satisfies the identity (2.4) by Theorem 2.3.1 in Chapter II.

It remains to establish the surjectivity of θ. Let x be an arbitrary element in $\mathcal{O}_{\mathbb{C}_K}$. Since $\mathcal{O}_{\mathbb{C}_K}$ is p-adically complete, it suffices to find a sequence (c_n) in $\mathcal{O}_F$ with

$$x - \sum_{n=0}^{m} c_n^\sharp p^n \in p^{m+1}\mathcal{O}_{\mathbb{C}_K} \quad \text{for each } m \geq 0.$$

In fact, we can use Lemma 2.1.9 to inductively construct such a sequence by setting each c_m to be an element in $\mathcal{O}_F$ with

$$\frac{1}{p^m}\left(x - \sum_{n=0}^{m-1} c_n^\sharp p^n\right) - c_m^\sharp \in p\mathcal{O}_{\mathbb{C}_K},$$

thereby completing the proof. ☐

Remark. Our proof remains valid if we replace $\mathbb{C}_K$ by an arbitrary perfectoid field C; in other words, every perfectoid field C yields a surjective ring homomorphism $\theta_C : W(\mathcal{O}_{C^\flat}) \twoheadrightarrow \mathcal{O}_C$.

Definition 2.2.5. We refer to the map θ in Proposition 2.2.4 as the *Fontaine map* and let $\theta[1/p] : A_{\mathrm{inf}}[1/p] \to \mathbb{C}_K$ denote the ring homomorphism induced by θ.

Remark. As explained by Brinon-Conrad [**BC**, Lemma 4.4.1], we can construct the Fontaine map θ without using Theorem 2.3.1 from Chapter II. In this approach, we first define θ as a set theoretic map given by the identity (2.4) and show that θ is indeed a ring homomorphism using descriptions of the ring operations on $A_{\mathrm{inf}} = W(\mathcal{O}_F)$.

PROPOSITION 2.2.6. The ring homomorphism $\theta[1/p] : A_{\mathrm{inf}}[1/p] \to \mathbb{C}_K$ is surjective.

PROOF. For every $c \in \mathbb{C}_K$, there exists an integer $n \geq 0$ with $p^n c \in \mathcal{O}_{\mathbb{C}_K}$. Hence the assertion immediately follows from Proposition 2.2.4. ☐

Definition 2.2.7. We define the *de Rham local ring* to be

$$B_{\mathrm{dR}}^{+} := \varprojlim_{i} A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p])^{i}$$

and let $\theta_{\mathrm{dR}}^{+} : B_{\mathrm{dR}}^{+} \twoheadrightarrow A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p])$ denote the natural projection.

Remark. We will soon define the de Rham period ring B_{dR} to be the fraction field of B_{dR}^{+} after verifying that B_{dR}^{+} is a discrete valuation ring. At this point, it is instructive to explain Fontaine's insight behind the construction of B_{dR}. As briefly discussed in Chapter I, Fontaine introduced the rings B_{HT} and B_{dR} respectively to formulate the Hodge-Tate decomposition and the de Rham comparison isomorphism. The de Rham cohomology of a proper smooth variety over K admits the Hodge filtration with the Hodge cohomology as its graded vector space. Fontaine sought B_{dR} as the fraction field of a complete discrete valuation ring B_{dR}^{+} with residue field $\mathbb{C}_K$ so that it admits a filtration $\{\, \mathrm{Fil}^n(B_{\mathrm{dR}}) \,\}_{n\in\mathbb{Z}} := \{\, t^n B_{\mathrm{dR}}^{+} \,\}_{n\in\mathbb{Z}}$ for a uniformizer $t \in B_{\mathrm{dR}}^{+}$ with its graded ring isomorphic to B_{HT}. For a perfect field k of characteristic p, the theory of Witt vectors yields a complete discrete valuation ring with residue field k by Lemma 2.3.9 in Chapter II. Fontaine judiciously adjusted the construction of Witt vectors for the field $\mathbb{C}_K$ of characteristic 0 by passing to characteristic p, or by tilting the perfectoid field $\mathbb{C}_K$ in modern language. He began by taking the ring $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ which is evidently of characteristic p. As $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$ turns out to be not perfect, Fontaine considered its *perfection* $\varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K} \cong \mathcal{O}_F$ by adding all p-power roots of elements in $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. Fontaine then discovered that $A_{\mathrm{inf}} = W(\mathcal{O}_F)$ gives rise to a surjective ring homomorphism $\theta[1/p] : A_{\mathrm{inf}}[1/p] \twoheadrightarrow \mathbb{C}_K$. Moreover, as we will soon see, $\ker(\theta[1/p])$ turned out to be a principal ideal. Therefore Fontaine obtained the desired ring B_{dR}^{+} as the completion of $A_{\mathrm{inf}}[1/p]$ with respect to $\ker(\theta[1/p])$.

LEMMA 2.2.8. For each integer $n \geq 0$ we have $\ker(\theta) \cap p^n A_{\mathrm{inf}} = p^n \ker(\theta)$.

PROOF. Since we evidently have $p^n \ker(\theta) \subseteq \ker(\theta) \cap p^n A_{\mathrm{inf}}$, we only need to show that every $a \in \ker(\theta) \cap p^n A_{\mathrm{inf}}$ is an element of $p^n \ker(\theta)$. Let us write $a = p^n b$ for some $b \in A_{\mathrm{inf}}$. From the identity

$$0 = \theta(a) = \theta(p^n b) = p^n \theta(b)$$

we find $\theta(b) = 0$ as $\mathcal{O}_{\mathbb{C}_K}$ is torsion free. Therefore we deduce that $a = p^n b$ lies in $p^n \ker(\theta)$ as desired. □

LEMMA 2.2.9. The sharp map associated to $\mathbb{C}_K$ is surjective.

PROOF. The field $\mathbb{C}_K$ is algebraically closed as noted in Chapter II, Proposition 3.1.10. Hence we deduce that the p-th power map on $\mathbb{C}_K$ is surjective and in turn obtain the desired assertion. □

Remark. It is worthwhile to mention that Lemma 2.2.9 is not essential for our discussion. In fact, we use Lemma 2.2.9 only to give a simple description of an element generating $\ker(\theta)$. For an arbitrary perfectoid field C, we can still show that the kernel of the surjective ring homomorphism $\theta_C : W(\mathcal{O}_{C^\flat}) \twoheadrightarrow \mathcal{O}_C$ is principal by explicitly presenting a generator.

Definition 2.2.10. A *distinguished element* of A_{inf} is an element of the form $\xi = [p^\flat] - p \in A_{\mathrm{inf}}$ for some $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$.

Remark. The existence of $p^\flat$ follows from Lemma 2.2.9. We may regard $p^\flat$ as a system of p-power roots of p in $\mathbb{C}_K$.

For the rest of this chapter, we fix a distinguished element $\xi = [p^\flat] - p \in A_{\text{inf}}$.

Lemma 2.2.11. Every element $a \in \ker(\theta)$ is an A_{inf}-linear combination of ξ and p.

Proof. We wish to show that a lies in the ideal generated by ξ and p, or equivalently by $[p^\flat]$ and p. Let us write

$$a = \sum_{n \geq 0} [c_n] p^n = [c_0] + p \sum_{n \geq 1} [c_n] p^{n-1} \quad \text{with } c_n \in \mathcal{O}_F.$$

It suffices to show that $[c_0]$ is divisible by $[p^\flat]$. Since we have $0 = \theta(a) = \sum_{n \geq 0} c_n^\sharp p^n$, we deduce that $c_0^\sharp$ is divisible by p and consequently find

$$\nu^\flat(c_0) = \nu(c_0^\sharp) \geq \nu(p) = \nu((p^\flat)^\sharp) = \nu^\flat(p^\flat).$$

Hence there exists an element $u \in \mathcal{O}_F$ with $c_0 = p^\flat u$ or equivalently $[c_0] = [p^\flat][u]$. □

Proposition 2.2.12. The element $\xi \in A_{\text{inf}}$ generates the ideal $\ker(\theta)$ in A_{inf}.

Proof. The ideal $\ker(\theta)$ contains ξ as we have

$$\theta(\xi) = \theta([p^\flat] - p) = (p^\flat)^\sharp - p = p - p = 0.$$

Hence we only need to show that every $a \in \ker(\theta)$ lies in the ideal ξA_{inf}. Since A_{inf} is p-adically complete by construction, it suffices to present a sequence (c_n) in A_{inf} with

$$a - \sum_{n=0}^{m} c_n \xi p^n \in p^{m+1} A_{\text{inf}} \quad \text{for each } m \geq 0.$$

We take $c_0 \in A_{\text{inf}}$ with $a - c_0 \xi \in pA_{\text{inf}}$ given by Lemma 2.2.11 and inductively construct c_m for each $m \geq 1$. In fact, by Lemma 2.2.8 we have

$$a - \sum_{n=0}^{m-1} c_n \xi p^n \in \ker(\theta) \cap p^m A_{\text{inf}} = p^m \ker(\theta)$$

and thus find $b_m, c_m \in A_{\text{inf}}$ with

$$a - \sum_{n=0}^{m-1} c_n \xi p^n = p^m (p b_m + c_m \xi)$$

or equivalently

$$a - \sum_{n=0}^{m} c_n \xi p^n = p^{m+1} b_m$$

as desired. □

Remark. Proposition 2.2.12 yields a natural isomorphism $A_{\text{inf}}/\xi A_{\text{inf}} \cong \mathcal{O}_{\mathbb{C}_K}$, which turns out to be a topological isomorphism. Since the construction of A_{inf} depends only on the field F, the principal ideal $\xi A_{\text{inf}} \subseteq A_{\text{inf}}$ contains all necessary information for recovering the perfectoid field $\mathbb{C}_K$ from its tilt F. In fact, we will see in Chapter IV that every perfectoid field C with $C^\flat \simeq F$ arises as the fraction field of A_{inf}/I for a unique principal ideal $I \subseteq A_{\text{inf}}$.

Proposition 2.2.13. The element $\xi \in A_{\text{inf}}$ generates the ideal $\ker(\theta[1/p])$ in $A_{\text{inf}}[1/p]$.

Proof. For every $a \in \ker(\theta[1/p])$, we have $p^n a \in \ker(\theta)$ for some $n > 0$. Hence the assertion follows from Proposition 2.2.12. □

Lemma 2.2.14. Every $a \in A_{\text{inf}}[1/p]$ with $\xi a \in A_{\text{inf}}$ is an element in A_{inf}.

Proof. Since we have $\theta(\xi a) = \theta[1/p](\xi a) = 0$ by Proposition 2.2.13, we find $\xi a \in \xi A_{\text{inf}}$ by Proposition 2.2.12 and in turn apply Lemma 1.1.8 to see that a lies in A_{inf}. □

Lemma 2.2.15. For each integer $i \geq 1$, we have $A_{\text{inf}} \cap \ker(\theta[1/p])^i = \ker(\theta)^i$.

Proof. Since we clearly have $\ker(\theta)^i \subseteq A_{\text{inf}} \cap \ker(\theta[1/p])^i$, we only need to show that every $a \in A_{\text{inf}} \cap \ker(\theta[1/p])^i$ lies in $\ker(\theta)^i$. Proposition 2.2.13 yields an element $b \in A_{\text{inf}}[1/p]$ with $a = \xi^i b$. Hence we find $b \in A_{\text{inf}}$ by Lemma 2.2.14 and consequently deduce the desired assertion from Proposition 2.2.12. □

Proposition 2.2.16. We have $\displaystyle\bigcap_{i=1}^{\infty} \ker(\theta)^i = \bigcap_{i=1}^{\infty} \ker(\theta[1/p])^i = 0$.

Proof. By Lemma 2.2.15 we have

$$\bigcap_{i=1}^{\infty} \ker(\theta[1/p])^i = \left(\bigcap_{i=1}^{\infty} \ker(\theta)^i \right) [1/p].$$

Hence it suffices to establish the identity $\displaystyle\bigcap_{i=1}^{\infty} \ker(\theta)^i = 0$. Let us take an arbitrary element $c \in \displaystyle\bigcap_{i=1}^{\infty} \ker(\theta)^i$ and write $c = \sum [c_n] p^n$ with $c_n \in \mathcal{O}_F$. Proposition 2.2.12 shows that c is divisible by every power of $\xi = [p^\flat] - p$ in A_{inf}, which in particular implies that c_0 is divisible by every power of $p^\flat$ in $\mathcal{O}_F$. Since we have $\nu^\flat(p^\flat) = \nu((p^\flat)^\sharp) = \nu(p) = 1 > 0$, we find $c_0 = 0$ and in turn write $c = pc'$ for some $c' \in A_{\text{inf}}$. Moreover, Lemma 2.2.15 yields the relation

$$c' \in A_{\text{inf}} \cap \left(\bigcap_{i=1}^{\infty} \ker(\theta)^i \right) [1/p] = A_{\text{inf}} \cap \left(\bigcap_{i=1}^{\infty} \ker(\theta[1/p])^i \right) = \bigcap_{i=1}^{\infty} \ker(\theta)^i.$$

Now a simple induction shows that c is infinitely divisible by p and thus is zero. □

Proposition 2.2.17. The ring B_{dR}^+ is a complete discrete valuation ring with $\ker(\theta_{\text{dR}}^+)$ as the maximal ideal, $\mathbb{C}_K$ as the residue field, and ξ as a uniformizer.

Proof. Since we have $B_{\text{dR}}^+/\ker(\theta_{\text{dR}}^+) \cong \mathbb{C}_K$ by Proposition 2.2.6, we deduce from some general facts stated in the Stacks project [**Sta**, Tag 05GI and Tag 00E9] that B_{dR}^+ is a local ring with $\ker(\theta_{\text{dR}}^+)$ as the maximal ideal and $\mathbb{C}_K$ as the residue field. Let us now consider an arbitrary nonzero element $b \in B_{\text{dR}}^+$. For each integer $i \geq 0$, we write b_i and ξ_i respectively for the images of b and ξ under the projection $B_{\text{dR}}^+ \twoheadrightarrow A_{\text{inf}}[1/p]/\ker(\theta[1/p])^i$. In addition, we take the largest integer $j \geq 0$ with $b_j = 0$. Proposition 2.2.12 implies that for each $i > j$ we may write $b_i = \xi_i^j u_i$ with $u_i \notin \ker(\theta[1/p])/\ker(\theta[1/p])^i$. For each $i > j$ we let u_i' denote the image of u_i in $A_{\text{inf}}[1/p]/\ker(\theta[1/p])^{i-j}$. We observe that the sequence $(u_i')_{i>j}$ depends only on b and gives rise to a unique unit $u \in B_{\text{dR}}^+$ with $b = \xi^j u$. Therefore B_{dR}^+ is a discrete valuation ring with ξ as a uniformizer. Now we deduce from Proposition 2.2.12 and Proposition 2.2.16 that B_{dR}^+ is complete, thereby establishing the desired assertion. □

Remark. Our argument so far in this subsection remains valid if we replace $\mathbb{C}_K$ by an arbitrary algebraically closed perfectoid field of characteristic 0.

Definition 2.2.18. The *de Rham period ring* B_{dR} is the fraction field of B_{dR}^+.

Proposition 2.2.19. Let K_0 denote the fraction field of $W(k)$.

(1) The field K is a finite totally ramified extension of K_0.
(2) There exists a natural commutative diagram

$$\begin{array}{ccc} K_0 & \hookrightarrow & A_{\inf}[1/p] \\ \downarrow & & \downarrow \\ \overline{K} & \hookrightarrow & B_{\mathrm{dR}}^+ \\ & \searrow & \downarrow \theta_{\mathrm{dR}}^+ \\ & & \mathbb{C}_K \end{array}$$

where the diagonal map is the natural inclusion.

Proof. Let us take a uniformizer π of $\mathcal{O}_K$. There exists an integer $e > 0$ with $p = \pi^e u$ for some unit $u \in \mathcal{O}_K$. Hence we obtain a natural ring homomorphism

$$k = \mathcal{O}_K/\pi\mathcal{O}_K \longrightarrow \mathcal{O}_K/\pi^e\mathcal{O}_K = \mathcal{O}_K/p\mathcal{O}_K \tag{2.5}$$

which identifies $\mathcal{O}_K/p\mathcal{O}_K$ as a k-algebra with a basis given by $1, \pi, \cdots, \pi^{e-1}$. The map (2.5) induces a ring homomorphism $W(k) \to \mathcal{O}_K$ by Theorem 2.3.1 in Chapter II.

We assert that $1, \pi, \cdots, \pi^{e-1}$ generate $\mathcal{O}_K$ over $W(k)$. Take an arbitrary element $c \in \mathcal{O}_K$. Since $\mathcal{O}_K$ is p-adically complete, it suffices to find sequences $(a_{0,n}), \cdots, (a_{e-1,n})$ in $W(k)$ with

$$c - \sum_{i=0}^{e-1}\sum_{n=0}^{m} a_{i,n}p^n\pi^i \in p^{m+1}\mathcal{O}_K \quad \text{for each } m \geq 0.$$

In fact, we use the map (2.5) to inductively obtain $a_{0,m}, \cdots, a_{e-1,m} \in W(k)$ with

$$\frac{1}{p^m}\left(c - \sum_{i=0}^{e-1}\sum_{n=0}^{m-1} a_{i,n}p^n\pi^i\right) - \sum_{i=0}^{e-1} a_{i,m}\pi^i \in p\mathcal{O}_K$$

and consequently obtain the desired assertion.

Our discussion in the previous paragraph shows that K is a finite extension of K_0 and in turn yields statement (1) as both K_0 and K have residue field k. Hence it remains to establish statement (2). The map (2.5) induces a ring homomorphism $k \to \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. Since k is perfect, this map gives rise to a natural homomorphism

$$k \longrightarrow \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K} \cong \mathcal{O}_F$$

with the isomorphism given by Proposition 2.1.7 and in turn yields the top horizontal map by Theorem 2.3.1 in Chapter II. Moreover, we get the left vertical map from statement (1) and take the right vertical map to be the natural map

$$A_{\inf}[1/p] \to \varprojlim_i A_{\inf}[1/p]/\ker(\theta[1/p])^i = B_{\mathrm{dR}}^+$$

which is injective by Proposition 2.2.16. We may now identify K_0 as a subring of B_{dR}^+. Statement (1) and Proposition 2.2.17 together show that $\overline{K}$ is a separable algebraic extension of K_0 which lies in the residue field $\mathbb{C}_K$ of the complete discrete valuation ring B_{dR}^+. Therefore Hensel's lemma implies that $\overline{K}$ admits a unique embedding into B_{dR}^+ which fits in the desired diagram. □

In order to study some additional properties of B_{dR}, we invoke the following technical result without a proof.

PROPOSITION 2.2.20. There exists a refinement of the discrete valuation topology on B_{dR}^+ with the following properties:

(i) The natural map $A_{\mathrm{inf}} \to B_{\mathrm{dR}}^+$ identifies A_{inf} as a closed subring of B_{dR}^+.
(ii) The map $\theta[1/p]$ is continuous and open with respect to the p-adic topology on $\mathbb{C}_K$.
(iii) There exists a continuous map $\log : \mathbb{Z}_p(1) \to B_{\mathrm{dR}}^+$ with

$$\log(c) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([c]-1)^n}{n} \quad \text{for every } c \in \mathbb{Z}_p(1)$$

under the natural identification $\mathbb{Z}_p(1) = \left\{ c \in \mathcal{O}_F : c^\sharp = 1 \right\}$.
(iv) The ring B_{dR}^+ is complete.

Remark. We will eventually prove Proposition 2.2.20 in Chapter IV after constructing the Fargues-Fontaine curve. There will be no circular reasoning as the construction of the Fargues-Fontaine curve relies only on results that we have discussed prior to Proposition 2.2.20. Curious readers may consult the notes of Brinon-Conrad [**BC**, Exercise 4.5.3] for a sketch of a proof which does not involve the Fargues-Fontaine curve.

Let us briefly explain why Proposition 2.2.20 is essential for our discussion. The discrete valuation topology on B_{dR}^+ has a major defect of not carrying much information about the p-adic topology on $\mathbb{C}_K$. In fact, if we only consider the discrete valuation topology on B_{dR}^+ the map $\theta[1/p]$ is not continuous with respect to the p-adic topology on $\mathbb{C}_K$. Proposition 2.2.20 allows us to incoorporates the p-adic topology on $\mathbb{C}_K$ in our discussion, which is pivotal for studying continuous Γ_K-representations.

Definition 2.2.21. We refer to the map $\log : \mathbb{Z}_p(1) \to B_{\mathrm{dR}}^+$ given by Proposition 2.2.20 as the *cyclotomic logarithm*.

LEMMA 2.2.22. Let ε be a basis element of $\mathbb{Z}_p(1) = \left\{ c \in \mathcal{O}_F : c^\sharp = 1 \right\}$ over $\mathbb{Z}_p$.

(1) The element ξ divides $[\varepsilon]-1$ in A_{inf}.
(2) We have $\nu^\flat(\varepsilon - 1) = \dfrac{p}{p-1}$.

PROOF. Since $[\varepsilon]-1$ satisfies the equality

$$\theta([\varepsilon]-1) = \varepsilon^\sharp - 1 = 1 - 1 = 0,$$

statement (1) follows from Proposition 2.2.12. Let us now write $\varepsilon = (\zeta_{p^n})$ where each ζ_{p^n} is a primitive p^n-th root of unity in $\overline{K}$. We use Proposition 2.1.7 and the continuity of ν to find

$$\nu^\flat(\varepsilon - 1) = \nu\left((\varepsilon-1)^\sharp\right) = \nu\left(\lim_{n\to\infty} (\zeta_{p^n}-1)^{p^n}\right) = \lim_{n\to\infty} p^n \nu(\zeta_{p^n}-1).$$

In addition, we note that the minimal polynomial of $\zeta_{p^n}-1$ over $\mathbb{Q}_p$ is $f(x) = \displaystyle\sum_{i=0}^{p-1} (x+1)^{ip^{n-1}}$ of degree $p^{n-1}(p-1)$ with constant term p. Since the roots of the irreducible polynomial $f(x)$ over $\mathbb{Q}_p$ have the same p-adic valuation, we obtain the equality

$$\nu(\zeta_{p^n}-1) = \frac{\nu(p)}{p^{n-1}(p-1)} = \frac{1}{p^{n-1}(p-1)}$$

and in turn establish statement (2). □

PROPOSITION 2.2.23. Let ε be a basis element of $\mathbb{Z}_p(1) = \left\{ c \in \mathcal{O}_F : c^\sharp = 1 \right\}$ over $\mathbb{Z}_p$.

(1) The element $t := \log(\varepsilon) \in B_{\text{dR}}^+$ is a uniformizer.
(2) For every $m \in \mathbb{Z}_p$, we have $\log(\varepsilon^m) = m \log(\varepsilon)$.

PROOF. Let us first consider statement (1). By Proposition 2.2.19 and Lemma 2.2.22, we have $[\varepsilon] - 1 \in \xi A_{\text{inf}}$ and $\dfrac{([\varepsilon]-1)^n}{n} \in \xi^2 B_{\text{dR}}^+$ for each $n \geq 2$. Hence we find

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} \in ([\varepsilon]-1) + \xi^2 B_{\text{dR}}^+.$$

Since ξ is a uniformizer of B_{dR}^+ as noted in Proposition 2.2.17, it suffices to show that $[\varepsilon] - 1$ is not divisible by ξ^2 in B_{dR}^+.

Suppose for contradiction that $[\varepsilon] - 1$ lies in $\xi^2 B_{\text{dR}}^+$. Proposition 2.2.17 shows that $[\varepsilon] - 1$ maps to 0 under the projection $B_{\text{dR}}^+ \twoheadrightarrow A_{\text{inf}}[1/p]/\ker(\theta[1/p])^2$. Hence Proposition 2.2.12 and Lemma 2.2.15 together imply that $[\varepsilon] - 1$ is divisible by ξ^2 in A_{inf}. Since the first terms in the Teichmüller expansions for $[\varepsilon] - 1$ and ξ^2 are respectively $[\varepsilon - 1]$ and $[(p^\flat)^2]$, we have

$$\nu^\flat(\varepsilon - 1) \geq \nu^\flat((p^\flat)^2) = 2\nu^\flat(p^\flat) = 2\nu((p^\flat)^\sharp) = 2\nu(p) = 2.$$

If p is odd, we find $\nu^\flat(\varepsilon - 1) < 2$ by Lemma 2.2.22 and in turn obtain a desired contradiction. For $p = 2$, we write $[\varepsilon] - 1 = \xi^2 a$ for some $a \in A_{\text{inf}}$ and compare the coefficients of p in the Teichmüller expansions using Proposition 2.3.7 from Chapter II to deduce the equality $\varepsilon - 1 = c_1^2 (p^\flat)^4$, where c_1 denotes the coefficient of p in the Teichmüller expansion of a. Hence for $p = 2$ we have

$$\nu^\flat(\varepsilon - 1) \geq \nu^\flat((p^\flat)^4) = 4\nu^\flat(p^\flat) = 4\nu((p^\flat)^\sharp) = 4\nu(p) = 4$$

and accordingly obtain a desired contradiction by Lemma 2.2.22.

It remains to establish statement (2). If m is an integer, we have

$$\log((1+x)^m) = m \log(1+x)$$

as formal power series and thus set $x = \varepsilon - 1$ to find $\log(\varepsilon^m) = m \log(\varepsilon)$. For the general case, let us choose a sequence (m_n) of integers with each $m_n - m$ divisible by p^n. It is straightforward to verify the equality

$$\lim_{n \to \infty} \varepsilon^{m_n} = \varepsilon^m,$$

for example by writing $\varepsilon = (\zeta_{p^n})$ with each ζ_{p^n} being a primitive p^n-th root of unity in $\overline{K}$. Hence we apply Proposition 2.2.20 to find

$$\log(\varepsilon^m) = \log\left(\lim_{n\to\infty} \varepsilon^{m_n}\right) = \lim_{n\to\infty} \log(\varepsilon^{m_n}) = \lim_{n\to\infty} m_n \log(\varepsilon) = m \log(\varepsilon),$$

thereby completing the proof. □

Remark. We can adjust our argument in the first paragraph to show that the power series for $\log(\varepsilon)$ converges under the discrete valuation topology on B_{dR}^+. Hence the topology given by Proposition 2.2.20 is not necessary for constructing the cyclotomic logarithm.

Definition 2.2.24. A *cyclotomic uniformizer* of B_{dR}^+ is an element of the form $t = \log(\varepsilon)$ for some basis element ε of $\mathbb{Z}_p(1)$.

PROPOSITION 2.2.25. A cyclotomic uniformizer of B_{dR}^+ is unique up to $\mathbb{Z}_p^\times$-multiple.

PROOF. The assertion is evident by Proposition 2.2.23. □

Theorem 2.2.26 (Fontaine [**Fon82**]). The ring B_{dR} admits a natural action of Γ_K with the following properties:

(i) The cyclotomic logarithm and θ_{dR}^+ are Γ_K-equivariant.

(ii) Given a cyclotomic uniformizer $t \in B_{\mathrm{dR}}^+$, we have $\gamma(t) = \chi(\gamma)t$ for every $\gamma \in \Gamma_K$.

(iii) Every cyclotomic uniformizer $t \in B_{\mathrm{dR}}^+$ yields a natural Γ_K-equivariant isomorphism

$$\bigoplus_{n\in\mathbb{Z}} t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+ \cong \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_K(n) = B_{\mathrm{HT}}.$$

(iv) B_{dR} is $(\mathbb{Q}_p, \Gamma_K)$-regular with a canonical identification $B_{\mathrm{dR}}^{\Gamma_K} \cong K$.

Proof. Let us first describe the natural action of Γ_K on B_{dR}. The action of Γ_K on $\mathbb{C}_K$ naturally induces an action on $F = \varprojlim\limits_{x\mapsto x^p} \mathbb{C}_K$ as the p-th power map on $\mathbb{C}_K$ is Γ_K-equivariant. In fact, given an arbitrary element $x = (x_n) \in F$ we have $\gamma(x) = (\gamma(x_n))$ for every $\gamma \in \Gamma_K$. Since $\mathcal{O}_F$ is stable under the action of Γ_K, we apply Theorem 2.3.1 in Chapter II to obtain a natural action of Γ_K on $A_{\mathrm{inf}}[1/p]$ with

$$\gamma\left(\sum [c_n]p^n\right) = \sum [\gamma(c_n)]p^n \quad \text{ for each } \gamma \in \Gamma_K \text{ and } c_n \in \mathcal{O}_F.$$

Now we find that θ and $\theta[1/p]$ are both Γ_K-equivariant by construction, whicn in particular implies that both $\ker(\theta)$ and $\ker(\theta[1/p])$ are stable under the action of Γ_K. Hence Γ_K naturally acts on $B_{\mathrm{dR}}^+ = \varprojlim_i A_{\mathrm{inf}}[1/p]/\ker(\theta[1/p])^i$ and its fraction field B_{dR}.

With our discussion in the preceding paragraph, property (i) is straightforward to verify. Moreover, property (i) and Proposition 2.2.23 togther show that every $\gamma \in \Gamma_K$ acts on a cyclotomic uniformizer $t = \log(\varepsilon) \in B_{\mathrm{dR}}^+$ by the relation

$$\gamma(t) = \gamma(\log(\varepsilon)) = \log(\gamma(\varepsilon)) = \log(\varepsilon^{\chi(\gamma)}) = \chi(\gamma)\log(\varepsilon) = \chi(\gamma)t$$

and thus yield property (ii). Now we note by property (i) that the natural isomorphism

$$B_{\mathrm{dR}}^+/tB_{\mathrm{dR}}^+ = B_{\mathrm{dR}}^+/\ker(\theta_{\mathrm{dR}}^+) \cong \mathbb{C}_K$$

is Γ_K-equivariant and in turn obtain a Γ_K-equivariant isomorphism

$$t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+ \simeq \mathbb{C}_K(n) \quad \text{ for every } n \in \mathbb{Z}$$

by property (ii) and Lemma 3.1.3 in Chapter II. Since Proposition 2.2.25 shows that a cyclotomic uniformizer of B_{dR}^+ is unique up to $\mathbb{Z}_p^\times$-multiple, we deduce that this isomorphism is canonical and consequently establish property (iii).

It remains to verify property (iv). Example 1.1.2 shows that B_{dR} is $(\mathbb{Q}_p, \Gamma_K)$-regular for being a field extension of $\mathbb{Q}_p$. In addition, property (i) implies that the natural injective homomorphism $\overline{K} \hookrightarrow B_{\mathrm{dR}}^+$ given by Proposition 2.2.19 is Γ_K-equivariant and in turn induces an injective homomorphism

$$K = \overline{K}^{\Gamma_K} \longrightarrow (B_{\mathrm{dR}}^+)^{\Gamma_K} \longrightarrow B_{\mathrm{dR}}^{\Gamma_K}. \tag{2.6}$$

Now by property (iii) we get an injective K-algebra homomorphism

$$\bigoplus_{n\in\mathbb{Z}} (B_{\mathrm{dR}}^{\Gamma_K} \cap t^n B_{\mathrm{dR}}^+)/(B_{\mathrm{dR}}^{\Gamma_K} \cap t^{n+1} B_{\mathrm{dR}}^+) \longrightarrow B_{\mathrm{HT}}^{\Gamma_K}.$$

Since we have $B_{\mathrm{HT}}^{\Gamma_K} \cong K$ by Theorem 3.1.11 in Chapter II, the K-algebra on the source has dimension at most 1. Hence we find $\dim_K B_{\mathrm{dR}}^{\Gamma_K} \leq 1$ and in turn deduce that the map (2.6) is an isomorphism, thereby completing the proof. $\square$

2.3. Filtered vector spaces

In this subsection, we set up a categorical framework for our discussion of B_{dR}-admissible representations in the next subsection.

Definition 2.3.1. Let L be an arbitrary field.

(1) A *filtration* on a vector space V over L is a collection of subspaces $\{ \mathrm{Fil}^n(V) \}_{n\in\mathbb{Z}}$ with $\mathrm{Fil}^n(V) \supseteq \mathrm{Fil}^{n+1}(V)$ for every $n \in \mathbb{Z}$.

(2) A *filtered vector space* over L is an L-vector space V with a filtration $\{ \mathrm{Fil}^n(V) \}_{n\in\mathbb{Z}}$ that satisfies the relations $\bigcap\limits_{n\in\mathbb{Z}} \mathrm{Fil}^n(V) = 0$ and $\bigcup\limits_{n\in\mathbb{Z}} \mathrm{Fil}^n(V) = V$.

(3) A *graded vector space* over L is an L-vector space V with a decomposition $V = \bigoplus\limits_{n\in\mathbb{Z}} V_n$.

(4) For a filtered vector space V over L, its *associated graded vector space* is

$$\mathrm{gr}(V) := \bigoplus_{n\in\mathbb{Z}} \mathrm{gr}^n(V) \quad \text{with} \quad \mathrm{gr}^n(V) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Fil}^n(V)/\,\mathrm{Fil}^{n+1}(V).$$

Remark. Many authors do not require the relations $\bigcap\limits_{n\in\mathbb{Z}} \mathrm{Fil}^n(V) = 0$ and $\bigcup\limits_{n\in\mathbb{Z}} \mathrm{Fil}^n(V) = V$ for a filtered vector space V.

Example 2.3.2. We present some examples given by Proposition 1.1.10 and Theorem 2.2.26.

(1) The de Rham period ring B_{dR} is a filtered K-algebra with

$$\mathrm{Fil}^n(B_{\mathrm{dR}}) := t^n B_{\mathrm{dR}}^+ \quad \text{and} \quad \mathrm{gr}(B_{\mathrm{dR}}) \cong B_{\mathrm{HT}}$$

where t is a cyclotomic uniformizer of B_{dR}^+.

(2) Every $V \in \mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ naturally gives rise to a graded K-vector space

$$D_{\mathrm{HT}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{\Gamma_K} = \bigoplus_{n\in\mathbb{Z}} (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(n))^{\Gamma_K}$$

and a filtered K-vector space

$$D_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K} \quad \text{with} \quad \mathrm{Fil}^n(D_{\mathrm{dR}}(V)) := (V \otimes_{\mathbb{Q}_p} t^n B_{\mathrm{dR}}^+)^{\Gamma_K}.$$

Definition 2.3.3. Let L be an arbitrary field.

(1) Given filtered vector spaces V and W over L, an L-linear map $f : V \to W$ is *filtered* if it maps each $\mathrm{Fil}^n(V)$ into $\mathrm{Fil}^n(W)$.

(2) Given graded vector spaces $V = \bigoplus\limits_{n\in\mathbb{Z}} V_n$ and $W = \bigoplus\limits_{n\in\mathbb{Z}} W_n$ over L, an L-linear map $f : V \to W$ is *graded* if it maps each V_n into W_n.

Remark. A filtered isomorphism is a filtered bijection with a filtered inverse. Similarly, a graded isomorphism is a graded bijection with a graded inverse.

Example 2.3.4. As mentioned in Chapter I, every proper smooth variety X over K yields a canonical K-linear graded isomorphism

$$D_{\mathrm{HT}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K})$$

and a canonical K-linear filtered isomorphism

$$D_{\mathrm{dR}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{\mathrm{dR}}(X/K).$$

PROPOSITION 2.3.5. Let $V = \bigoplus_{n\in\mathbb{Z}} V_n$ be a graded vector space over a field L.

(1) Given a graded vector space $W = \bigoplus_{n\in\mathbb{Z}} W_n$ over L, the tensor product $V \otimes_L W$ is naturally a graded L-vector space with

$$V \otimes_L W = \bigoplus_{n\in\mathbb{Z}} \left(\bigoplus_{i+j=n} V_i \otimes_L W_j \right).$$

(2) The dual $V^\vee$ is naturally a graded L-vector space with $V^\vee = \bigoplus_{n\in\mathbb{Z}} V_{-n}^\vee$.

PROOF. The assertions are stragithforward to verify. □

PROPOSITION 2.3.6. Let V be a filtered vector space over a field L.

(1) Given a filtered L-vector space W, the tensor product $V \otimes_L W$ is naturally a filtered L-vector space with

$$\mathrm{Fil}^n(V \otimes_L W) = \sum_{i+j=n} \mathrm{Fil}^i(V) \otimes_L \mathrm{Fil}^j(W) \quad \text{for every } n \in \mathbb{Z}.$$

(2) The dual $V^\vee$ is naturally a filtered L-vector space with

$$\mathrm{Fil}^n(V^\vee) = \left\{ f \in V^\vee : \mathrm{Fil}^{1-n}(V) \subseteq \ker(f) \right\} \cong \left(V/\,\mathrm{Fil}^{1-n}(V)\right)^\vee \quad \text{for every } n \in \mathbb{Z}.$$

PROOF. The assertions are stragithforward to verify. □

Example 2.3.7. Every field L is canonically a filtered vector space over itself with

$$\mathrm{Fil}^n(L) = \begin{cases} L & \text{for } n \leq 0, \\ 0 & \text{for } n > 0. \end{cases}$$

Given a filtered vector space V over L, we find

$$\mathrm{Fil}^n(V \otimes_L L) = \sum_{i+j=n} \mathrm{Fil}^i(V) \otimes_L \mathrm{Fil}^j(L) \cong \sum_{i\geq n} \mathrm{Fil}^i(V) = \mathrm{Fil}^n(V) \quad \text{for every } n \in \mathbb{Z}$$

by Proposition 2.3.6 and consequently obtain canonical filtered isomorphisms

$$V \cong V \otimes_L L \cong L \otimes_L V.$$

Moreover, the natural linear bijection $L \cong L^\vee$ is a filtered isomorphism as Proposition 2.3.6 yields an identification

$$\mathrm{Fil}^n(L^\vee) \cong \left(L/\,\mathrm{Fil}^{1-n}(L)\right)^\vee \cong \begin{cases} L & \text{for } n \leq 0, \\ 0 & \text{for } n > 0. \end{cases}$$

PROPOSITION 2.3.8. Given a filtered vector space V over a field L, the natural L-linear bijection $V \cong (V^\vee)^\vee$ is a filtered isomorphism.

PROOF. For every $n \in \mathbb{Z}$, we apply Proposition 2.3.6 to find

$$V^\vee/\,\mathrm{Fil}^{1-n}(V^\vee) \cong V^\vee/\,(V/\,\mathrm{Fil}^n(V))^\vee \cong \mathrm{Fil}^n(V)^\vee$$

and in turn obtain an identification

$$\mathrm{Fil}^n\left((V^\vee)^\vee\right) \cong \left(V^\vee/\,\mathrm{Fil}^{1-n}(V^\vee)\right)^\vee \cong \mathrm{Fil}^n(V)$$

as desired. □

PROPOSITION 2.3.9. Let V be a filtered vector space over a field L.

(1) Given a finite dimensional filtered L-vector space W, there exists a natural graded isomorphism
$$\mathrm{gr}(V \otimes_L W) \cong \mathrm{gr}(V) \otimes_L \mathrm{gr}(W).$$
(2) The dual $V^\vee$ yields a natural graded isomorphism
$$\mathrm{gr}(V^\vee) \cong \mathrm{gr}(V)^\vee.$$

PROOF. Let us begin with statement (1). By Proposition 2.3.5, it suffices to establish a canonical identification
$$\mathrm{gr}^n(V \otimes_L W) \cong \bigoplus_{i+j=n} \mathrm{gr}^i(V) \otimes_L \mathrm{gr}^j(W) \quad \text{for every } n \in \mathbb{Z}. \tag{2.7}$$
Since W is finite dimensional, we have $\mathrm{Fil}^r(W) = W$ and $\mathrm{Fil}^s(W) = 0$ for some $r,\ s \in \mathbb{Z}$. Hence Proposition 2.3.6 yields a natural isomorphism
$$\mathrm{Fil}^n(V \otimes_L W) = \sum_{j=r}^{s} \mathrm{Fil}^{n-j}(V) \otimes_L \mathrm{Fil}^j(W) \quad \text{for every } n \in \mathbb{Z}.$$
We can construct a basis $(v_{i,i'})$ of V such that each $\mathrm{Fil}^m(V)$ with $n-s \leq m \leq n-r+1$ has a basis $(v_{i,i'})_{i\geq m}$; indeed, we may fix a basis for $\mathrm{Fil}^{n-r+1}(V)$ and inductively extend a basis for each $\mathrm{Fil}^m(V)$ to $\mathrm{Fil}^{m-1}(V)$. Similarly, we can find a basis $(w_{j,j'})$ of W such that each $\mathrm{Fil}^m(W)$ has a basis $(w_{j,j'})_{j\geq m}$. Let us denote the image of each $v_{i,i'}$ under the map $\mathrm{Fil}^i(V) \twoheadrightarrow \mathrm{gr}^i(V)$ by $\overline{v}_{i,i'}$ and the image of each $w_{j,j'}$ under the map $\mathrm{Fil}^j(W) \twoheadrightarrow \mathrm{gr}^j(W)$ by $\overline{w}_{j,j'}$. We obtain the isomorphism (2.7) by observing that both sides admit a basis $(\overline{v}_{i,i'} \otimes \overline{w}_{j,j'})_{i+j=n}$. It is straightforward to verify that this isomorphism does not depend on bases $(v_{i,i'})$ and $(w_{j,j'})$.

It remains to establish statement (2). For every $n \in \mathbb{Z}$ we apply Proposition 2.3.6 to find
$$\begin{aligned} \mathrm{gr}^n(V^\vee) &\cong \mathrm{Fil}^n(V^\vee)/\mathrm{Fil}^{n+1}(V^\vee) \cong \big(V/\mathrm{Fil}^{1-n}(V)\big)^\vee / \big(V/\mathrm{Fil}^{-n}(V)\big)^\vee \\ &\cong \big(\mathrm{Fil}^{-n}(V)/\mathrm{Fil}^{-n+1}(V)\big)^\vee \cong \mathrm{gr}^{-n}(V)^\vee. \end{aligned}$$
Hence we deduce the desired assertion from Proposition 2.3.5. □

PROPOSITION 2.3.10. Given a field L, a bijective L-linear filtered map $f : V \to W$ is a filtered isomorphism if and only if the induced map $\mathrm{gr}(f) : \mathrm{gr}(V) \to \mathrm{gr}(W)$ is bijective.

PROOF. If f is a filtered isomorphism, the induced map $\mathrm{gr}(f)$ is clearly a graded isomorphism. Conversely, let us henceforth assume that $\mathrm{gr}(f)$ is bijective. We wish to show that for every $n \in \mathbb{Z}$ the induced map $\mathrm{Fil}^n(f) : \mathrm{Fil}^n(V) \to \mathrm{Fil}^n(W)$ is an isomorphism. The bijectivity of f implies that each $\mathrm{Fil}^n(f)$ is injective. Hence it remains to show that each $\mathrm{Fil}^n(f)$ is surjective. For every $n \in \mathbb{Z}$ we obtain a commutative diagram
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Fil}^{n+1}(V) & \longrightarrow & \mathrm{Fil}^n(V) & \longrightarrow & \mathrm{gr}^n(V) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathrm{Fil}^{n+1}(f)} & & \big\downarrow{\scriptstyle \mathrm{Fil}^{n}(f)} & & \big\downarrow{\scriptstyle \mathrm{gr}^{n}(f)} & & \\ 0 & \longrightarrow & \mathrm{Fil}^{n+1}(W) & \longrightarrow & \mathrm{Fil}^n(W) & \longrightarrow & \mathrm{gr}^n(W) & \longrightarrow & 0 \end{array}$$
with exact rows. Since each $\mathrm{gr}^n(f)$ is bijective, the snake lemma implies that the inclusion $\mathrm{Fil}^{n+1}(W) \hookrightarrow \mathrm{Fil}^n(W)$ induces a canonical isomorphism $\mathrm{coker}(\mathrm{Fil}^n(f)) \cong \mathrm{coker}(\mathrm{Fil}^{n+1}(f))$. Moreover, every $w \in \mathrm{Fil}^n(W)$ lies in the image of $\mathrm{Fil}^m(V)$ for some $m < n$ by the surjectivity of f and thus has zero image in $\mathrm{coker}(\mathrm{Fil}^m(f)) \cong \mathrm{coker}(\mathrm{Fil}^n(f))$. Hence we deduce that each $\mathrm{coker}(\mathrm{Fil}^n(f))$ vanishes as desired. □

2.4. Properties of de Rham representations

For the rest of this chapter, we write Vect_K, Fil_K, and Grd_K respectively for the categories of K-vector spaces, filtered K-vector spaces, and graded K-vector spaces. In addition, we fix a cyclotomic uniformizer $t = \log(\varepsilon)$ of B_{dR}^+ for some basis element ε of $\mathbb{Z}_p(1)$.

Definition 2.4.1. Let V be a p-adic Γ_K-representation.

(1) We refer to $D_{\mathrm{HT}}(V)$ and $D_{\mathrm{dR}}(V)$ in Example 2.3.2 respectively as the *Hodge-Tate graded space* and the *de Rham filtered space* associated to V.
(2) We say that V is *de Rham* if it is B_{dR}-admissible.

Example 2.4.2. For every proper smooth variety X over K, the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is de Rham as briefly discussed in Chapter I.

Proposition 2.4.3. If a p-adic Γ_K-representation V is de Rham, it is Hodge-Tate with a natural K-linear graded isomorphism

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{HT}}(V).$$

Proof. For every $n \in \mathbb{Z}$ we have a short exact sequence

$$0 \longrightarrow t^{n+1}B_{\mathrm{dR}}^+ \longrightarrow t^n B_{\mathrm{dR}}^+ \longrightarrow t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+ \longrightarrow 0,$$

which induces an exact sequence

$$0 \longrightarrow \left(V \otimes_{\mathbb{Q}_p} t^{n+1}B_{\mathrm{dR}}^+\right)^{\Gamma_K} \longrightarrow \left(V \otimes_{\mathbb{Q}_p} t^n B_{\mathrm{dR}}^+\right)^{\Gamma_K} \longrightarrow \left(V \otimes_{\mathbb{Q}_p} (t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+)\right)^{\Gamma_K}$$

and in turn yields an injective K-linear map

$$\mathrm{gr}^n(D_{\mathrm{dR}}(V)) = \mathrm{Fil}^n(D_{\mathrm{dR}}(V))/\,\mathrm{Fil}^{n+1}(D_{\mathrm{dR}}(V)) \hookrightarrow \left(V \otimes_{\mathbb{Q}_p} (t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+)\right)^{\Gamma_K}.$$

Hence we obtain an injective K-linear graded map

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \hookrightarrow \bigoplus_{n\in\mathbb{Z}} \left(V \otimes_{\mathbb{Q}_p} (t^n B_{\mathrm{dR}}^+/t^{n+1}B_{\mathrm{dR}}^+)\right)^{\Gamma_K} \cong (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{\Gamma_K} = D_{\mathrm{HT}}(V)$$

with the first identification given by Theorem 2.2.26. Moreover, we find

$$\dim_K D_{\mathrm{dR}}(V) = \dim_K \mathrm{gr}(D_{\mathrm{dR}}(V)) \leq \dim_K D_{\mathrm{HT}}(V) \leq \dim_{\mathbb{Q}_p} V$$

where the last inequality follows from Theorem 1.2.1. Since V is de Rham, both inequalities are indeed equalities and thus yield the desired assertion. $\square$

Proposition 2.4.4. Given a de Rham Γ_K-representation V, we have $\mathrm{gr}^n(D_{\mathrm{dR}}(V)) \neq 0$ if and only if n is a Hodge-Tate weight of V.

Proof. The assertion is an immediate consequence of Proposition 2.4.3. $\square$

Remark. Proposition 2.4.4 shows that Hodge-Tate weights of V under our sign convention coincide with the locations of jumps in the filtration of $D_{\mathrm{dR}}(V)$.

Example 2.4.5. Every Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ is de Rham; indeed, the inequality

$$\dim_K D_{\mathrm{dR}}(\mathbb{Q}_p(n)) \leq \dim_{\mathbb{Q}_p} \mathbb{Q}_p(n) = 1$$

given by Theorem 1.2.1 is an equality, as $D_{\mathrm{dR}}(\mathbb{Q}_p(n)) = (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}$ contains a nonzero element $1 \otimes t^{-n}$ by Theorem 2.2.26. In addition, $\mathbb{Q}_p(n)$ has a unique Hodge-Tate weight $-n$ as noted in Example 1.1.15. Hence Proposition 2.4.4 yields an identification

$$\mathrm{Fil}^m(D_{\mathrm{dR}}(\mathbb{Q}_p(n))) \cong \begin{cases} K & \text{for } m \leq -n, \\ 0 & \text{for } m > -n. \end{cases}$$

For the rest of this subsection, we denote by $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$ the category of de Rham Γ_K-representations.

LEMMA 2.4.6. Given an integer n, a p-adic Γ_K-representation V is de Rham if and only if its Tate twist $V(n)$ is de Rham.

PROOF. Since we have $V(n) \cong V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$ and $V \cong V(n) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-n)$, the assertion follows from Proposition 1.2.4 and Example 2.4.5. □

Example 2.4.7. Let V be an extension of $\mathbb{Q}_p(m)$ by $\mathbb{Q}_p(n)$ with $m < n$. We assert that V is de Rham. By Lemma 2.4.6, we may assume the equality $m = 0$ so that V fits into a short exact sequence

$$0 \longrightarrow \mathbb{Q}_p(n) \longrightarrow V \longrightarrow \mathbb{Q}_p \longrightarrow 0. \tag{2.8}$$

The functor D_{dR} is left exact by construction and thus yields an exact sequence

$$0 \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_p(n)) \longrightarrow D_{\mathrm{dR}}(V) \longrightarrow D_{\mathrm{dR}}(\mathbb{Q}_p).$$

We wish to establish the identity $\dim_K D_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V = 2$. Since we have

$$\dim_K D_{\mathrm{dR}}(\mathbb{Q}_p(n)) = \dim_K D_{\mathrm{dR}}(\mathbb{Q}_p) = 1$$

by Example 2.4.5, it suffices to show the surjectivity of the map $D_{\mathrm{dR}}(V) \to D_{\mathrm{dR}}(\mathbb{Q}_p) \cong K$.

The sequence (2.8) gives rise to a short exact sequence

$$0 \longrightarrow \mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ \longrightarrow \mathbb{Q}_p \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+ \longrightarrow 0.$$

In addition, Theorem 2.2.26 yields natural identifications

$$(\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \cong (t^n B_{\mathrm{dR}}^+)^{\Gamma_K} = 0 \quad \text{and} \quad (\mathbb{Q}_p \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \cong (B_{\mathrm{dR}}^+)^{\Gamma_K} \cong K.$$

Hence we obtain a long exact sequence

$$0 \longrightarrow 0 \longrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \longrightarrow K \longrightarrow H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+).$$

Since we have $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^+)^{\Gamma_K} \subseteq D_{\mathrm{dR}}(V)$, it is enough to prove that $H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+)$ vanishes.

By Theorem 2.2.26 we have a short exact sequence

$$0 \longrightarrow t^{n+1} B_{\mathrm{dR}}^+ \longrightarrow t^n B_{\mathrm{dR}}^+ \longrightarrow \mathbb{C}_K(n) \longrightarrow 0,$$

which in turn induces a long exact sequence

$$\mathbb{C}_K(n)^{\Gamma_K} \longrightarrow H^1(\Gamma_K, t^{n+1} B_{\mathrm{dR}}^+) \longrightarrow H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+) \longrightarrow H^1(\Gamma_K, \mathbb{C}_K(n)).$$

Now Theorem 3.1.11 in Chapter II implies that there exists an identification

$$H^1(\Gamma_K, t^{n+1} B_{\mathrm{dR}}^+) \cong H^1(\Gamma_K, t^n B_{\mathrm{dR}}^+). \tag{2.9}$$

Hence by induction we only need to show that $H^1(\Gamma_K, t B_{\mathrm{dR}}^+)$ vanishes.

Take an arbitrary cocycle $\alpha_0 : \Gamma_K \to t B_{\mathrm{dR}}^+$. For each $i \geq 1$, we use the identification (2.9) to inductively construct a cocycle $\alpha_i : \Gamma_K \to t^{i+1} B_{\mathrm{dR}}^+$ and an element $b_i \in t^i B_{\mathrm{dR}}^+$ with

$$\alpha_i(\gamma) = \alpha_{i-1}(\gamma) + \gamma(b_i) - b_i \quad \text{for every } \gamma \in \Gamma_K.$$

Since t is a uniformizer in B_{dR}^+, we take $b = \sum b_i \in B_{\mathrm{dR}}^+$ and find

$$\alpha_0(\gamma) + \gamma(b) - b = 0 \quad \text{for every } \gamma \in \Gamma_K.$$

We deduce that α_0 has trivial image in $H^1(\Gamma_K, t B_{\mathrm{dR}}^+)$, thereby completing the proof.

Remark. It is a highly nontrivial fact that every non-splitting extension of $\mathbb{Q}_p(1)$ by $\mathbb{Q}_p$ is not de Rham, even though it is Hodge-Tate as noted in Example 1.1.12.

PROPOSITION 2.4.8. Every de Rham Γ_K-representation V admits a natural Γ_K-equivariant K-linear filtered isomorphism

$$D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}.$$

PROOF. Since V is de Rham, Theorem 1.2.1 implies that the natural B_{dR}-linear map

$$D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \longrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \otimes_K B_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{dR}} \otimes_K B_{\mathrm{dR}}) \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

is Γ_K-equivariant and bijective. Moreover, this map is filtered as each arrow is evidently filtered. Now by Proposition 2.3.10, it suffices to show the bijectivity of the induced map

$$\mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}) \longrightarrow \mathrm{gr}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}). \tag{2.10}$$

Proposition 2.4.3 shows that V is Hodge-Tate with a natural isomorphism

$$\mathrm{gr}(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{HT}}(V).$$

We apply Theorem 2.2.26 and Proposition 2.3.9 to obtain canonical isomorphisms

$$\mathrm{gr}(D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}}) \cong \mathrm{gr}(D_{\mathrm{dR}}(V)) \otimes_K \mathrm{gr}(B_{\mathrm{dR}}) \cong D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}},$$
$$\mathrm{gr}(V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}) \cong V \otimes_{\mathbb{Q}_p} \mathrm{gr}(B_{\mathrm{dR}}) \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}.$$

Hence we identify the map (2.10) with the natural map

$$D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}} \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}$$

given by Theorem 1.2.1 and in turn deduce the desired assertion from Proposition 2.4.3. □

Remark. In our proof of Proposition 2.4.8, the finiteness of $\dim_{\mathbb{Q}_p}(V)$ and $\dim_K(D_{\mathrm{dR}}(V))$ are crucial for applying Proposition 2.3.9.

PROPOSITION 2.4.9. The functor D_{dR} with values in Fil_K is faithful and exact on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$.

PROOF. Since the forgetful functor $\mathrm{Fil}_K \to \mathrm{Vect}_K$ is faithful, Proposition 1.2.2 implies that D_{dR} is faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$. Hence it remains to verify that D_{dR} is exact on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K)$. Consider an exact sequence of de Rham Γ_K-representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0. \tag{2.11}$$

For every $n \in \mathbb{Z}$, we have an exact sequence

$$0 \longrightarrow \mathrm{Fil}^n(D_{\mathrm{dR}}(U)) \longrightarrow \mathrm{Fil}^n(D_{\mathrm{dR}}(V)) \longrightarrow \mathrm{Fil}^n(D_{\mathrm{dR}}(W)). \tag{2.12}$$

We wish to show that this sequence extends to a short exact sequence. Proposition 1.2.2 implies that the sequence (2.11) gives rise to a short exact sequence of K-vector spaces

$$0 \longrightarrow D_{\mathrm{HT}}(U) \longrightarrow D_{\mathrm{HT}}(V) \longrightarrow D_{\mathrm{HT}}(W) \longrightarrow 0.$$

It is straightforward to verify that this sequence is indeed a short exact sequence in Grd_K. Therefore Proposition 2.4.3 yields a short exact sequence of graded K-vector spaces

$$0 \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(U)) \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(V)) \longrightarrow \mathrm{gr}(D_{\mathrm{dR}}(W)) \longrightarrow 0.$$

Now for every $n \in \mathbb{Z}$ we find

$$\begin{aligned}\dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(V)) &= \sum_{i \geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(V)) \\ &= \sum_{i \geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(U)) + \sum_{i \geq n} \dim_K \mathrm{gr}^i(D_{\mathrm{dR}}(W)) \\ &= \dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(U)) + \dim_K \mathrm{Fil}^n(D_{\mathrm{dR}}(W))\end{aligned}$$

and in turn deduce that the sequence (2.12) extends to a short exact sequence as desired. □

PROPOSITION 2.4.10. Given a de Rham Γ_K-representation V, every subquotient W of V is de Rham with $D_{\text{dR}}(W)$ naturally identified as a subquotient of $D_{\text{dR}}(V)$ in Fil_K.

PROOF. The assertion is evident by Proposition 1.2.3 and Proposition 2.4.9. □

PROPOSITION 2.4.11. Given two de Rham Γ_K-representations V and W, their tensor product $V \otimes_{\mathbb{Q}_p} W$ is de Rham with a natural K-linear filtered isomorphism

$$D_{\text{dR}}(V) \otimes_K D_{\text{dR}}(W) \cong D_{\text{dR}}(V \otimes_{\mathbb{Q}_p} W). \tag{2.13}$$

PROOF. Proposition 1.2.4 shows that $V \otimes_{\mathbb{Q}_p} W$ is de Rham and yields the desired isomorphism (2.13) as a K-linear bijection. Since the construction of the map (2.13) rests on the multiplicative of B_{dR}, it is straightforward to verify that the map (2.13) is filtered. Moreover, we apply Proposition 2.4.3 and Proposition 2.3.9 to identify the induced map

$$\text{gr}(D_{\text{dR}}(V) \otimes_K D_{\text{dR}}(W)) \longrightarrow \text{gr}(D_{\text{dR}}(V \otimes_{\mathbb{Q}_p} W)).$$

with the natural isomorphism

$$D_{\text{HT}}(V) \otimes_K D_{\text{HT}}(W) \cong D_{\text{HT}}(V \otimes_{\mathbb{Q}_p} W)$$

given by Proposition 1.2.4. Now we deduce from Proposition 2.3.10 that the map (2.13) is a K-linear filtered isomorphism, thereby completing the proof. □

Example 2.4.12. Given a de Rham Γ_K-representation V, we have

$$\text{Fil}^m(D_{\text{dR}}(V(n))) \cong \text{Fil}^{m+n}(D_{\text{dR}}(V)) \quad \text{for each } m,\ n \in \mathbb{Z}$$

by Proposition 2.3.6, Example 2.4.5, and Proposition 2.4.11.

PROPOSITION 2.4.13. Given a de Rham Γ_K-representation V and a positive integer n, both $\wedge^n(V)$ and $\text{Sym}^n(V)$ are de Rham with natural K-linear filtered isomorphisms

$$\wedge^n(D_{\text{dR}}(V)) \cong D_{\text{dR}}(\wedge^n(V)) \quad \text{and} \quad \text{Sym}^n(D_{\text{dR}}(V)) \cong D_{\text{dR}}(\text{Sym}^n(V)).$$

PROOF. Proposition 1.2.5 shows that both $\wedge^n(V)$ and $\text{Sym}^n(V)$ are de Rham. Moreover, Proposition 1.2.5 yields the desired isomorphisms as K-linear bijections. Proposition 2.4.10 and Proposition 2.4.11 together imply that these maps are filtered isomorphisms. □

PROPOSITION 2.4.14. For every de Rham Γ_K-representation V, the dual representation $V^\vee$ is de Rham with a natural K-linear filtered perfect paring

$$D_{\text{dR}}(V) \otimes_K D_{\text{dR}}(V^\vee) \cong D_{\text{dR}}(V \otimes_{\mathbb{Q}_p} V^\vee) \longrightarrow D_{\text{dR}}(\mathbb{Q}_p) \cong K.$$

PROOF. Proposition 1.2.7 shows that $V^\vee$ is de Rham and yields the desired pairing as a K-linear perfect pairing. This pairing is filtered by Proposition 2.4.11 and thus gives rise to a filtered K-linear bijection

$$D_{\text{dR}}(V)^\vee \longrightarrow D_{\text{dR}}(V^\vee). \tag{2.14}$$

Moreover, we apply Proposition 2.3.9 and Proposition 2.4.3 to identify the induced map

$$\text{gr}(D_{\text{dR}}(V)^\vee) \longrightarrow \text{gr}(D_{\text{dR}}(V^\vee))$$

with the natural isomorphism

$$D_{\text{HT}}(V)^\vee \cong D_{\text{HT}}(V^\vee)$$

given by Proposition 1.2.7. Now we deduce from Proposition 2.3.10 that the map (2.14) is a K-linear filtered isomorphism, thereby completing the proof. □

Remark. Proposition 2.3.8 and Proposition 2.4.14 together show that the canonical isomorphism $V \cong (V^\vee)^\vee$ induces a natural K-linear filtered isomorphism $D_{\text{dR}}(V) \cong (D_{\text{dR}}(V)^\vee)^\vee$.

Definition 2.4.15. Given an extension L of K with an action of a group Γ, a *semilinear* Γ*-module* over L is an L-vector space M which carries a continuous Γ-action with

$$\gamma(cm) = \gamma(c)\gamma(m) \quad \text{for each } \gamma \in \Gamma,\ c \in L, \text{ and } m \in M.$$

LEMMA 2.4.16. Let L be a finite extension of K.

(1) L is naturally a p-adic field.
(2) If L is Galois over K, every semilinear $\mathrm{Gal}(L/K)$-module M over L admits a canonical isomorphism
$$M \cong M^{\mathrm{Gal}(L/K)} \otimes_K L.$$

PROOF. Statement (1) is straightforward to verify. For statement (2), let us now assume that L is Galois over K. Denote by $\mathrm{GalMod}_{L/K}$ the category of semilinear $\mathrm{Gal}(L/K)$-modules, where morphisms are $\mathrm{Gal}(L/K)$-equivariant L-linear maps. A general fact stated in the Stacks Project [**Sta**, Tag 0CDR] yields an equivalence

$$\mathrm{GalMod}_{L/K} \cong \mathrm{Vect}_K$$

which sends each $M \in \mathrm{GalMod}_{L/K}$ to $M^{\mathrm{Gal}(L/K)}$ with the inverse sending each $V \in \mathrm{Vect}_K$ to $V \otimes_K L$. Hence we establish the desired assertion. ☐

Remark. Lemma 2.4.16 admits an analogue for p-adic completion $\widehat{K^{\mathrm{un}}}$ of the maximal unramified extension K^{un} of K; indeed, $\widehat{K^{\mathrm{un}}}$ is a p-adic field with a natural Γ_k-action such that every semilinear Γ_k-module M over $\widehat{K^{\mathrm{un}}}$ admits a canonical isomorphism $M \cong M^{\Gamma_k} \otimes_K \widehat{K^{\mathrm{un}}}$.

PROPOSITION 2.4.17. Let V be a p-adic Γ_K-representation and L be a finite extension of K.

(1) There exists a natural L-linear filtered isomorphism
$$D_{\mathrm{dR},K}(V) \otimes_K L \cong D_{\mathrm{dR},L}(V)$$
where we set $D_{\mathrm{dR},K}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K}$ and $D_{\mathrm{dR},L}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_L}$.
(2) V is de Rham if and only if it is de Rham as a p-adic Γ_L-representation.

PROOF. Lemma 2.4.16 shows that L and its Galois closure L' in $\overline{K}$ are p-adic fields. If we set $D_{\mathrm{dR},L'}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_{L'}}$, we have

$$D_{\mathrm{dR},K}(V) \otimes_K L = (D_{\mathrm{dR},K}(V) \otimes_K L')^{\mathrm{Gal}(L'/L)} \quad \text{and} \quad D_{\mathrm{dR},L}(V) = D_{\mathrm{dR},L'}(V)^{\mathrm{Gal}(L'/L)}.$$

Hence we may replace L by L' to assume that L is Galois over K. Now we find

$$\mathrm{Fil}^n(D_{\mathrm{dR},K}(V)) = \mathrm{Fil}^n(D_{\mathrm{dR},L}(V))^{\mathrm{Gal}(L/K)} \quad \text{for every } n \in \mathbb{Z}$$

and in turn obtain statement (1) by Lemma 2.4.16. Statement (2) is an immediate consequence of statement (1). ☐

Remark. We can extend Proposition 2.4.17 to every p-adic field L with $K \subseteq L \subseteq \mathbb{C}_K$ by the remark following Lemma 2.4.16. Hence every $\mathbb{C}_K$-admissible Γ_K-representation is de Rham by a result of Sen [**Sen80**] stated after Theorem 1.1.7.

Example 2.4.18. Given a continuous character $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ with finite image, the corresponding Γ_K-representation $\mathbb{Q}_p(\eta)$ is de Rham with a K-linear filtered isomorphism

$$D_{\mathrm{dR}}(\mathbb{Q}_p(\eta)) \cong K \cong D_{\mathrm{dR}}(\mathbb{Q}_p). \tag{2.15}$$

In fact, if we take a finite extension L of K with $\Gamma_L \subseteq \ker(\eta)$, Proposition 2.4.17 yields an L-linear filtered isomorphism $D_{\mathrm{dR}}(\mathbb{Q}_p(\eta)) \otimes_K L \cong L$ which induces to the isomorphism (2.15).

Remark. Example 2.4.18 shows that $D_{\mathrm{dR}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{dR}}(\Gamma_K) \to \mathrm{Fil}_K$ is not fully faithful.

We close this section by introducing the *Fontaine-Mazur conjecture*, which proposes a classification of global p-adic representations arising from algebraic geometry.

CONJECTURE 2.4.19 (Fontaine-Mazur [**FM95**]). Let E be a number field and denote by $\mathcal{O}_E$ its ring of integers. An irreducible p-adic Γ_E-representation V is a subquotient of $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}}, \mathbb{Q}_p(m))$ for some proper smooth E-variety X if and only if it satisfies the following properties:

(i) V is unramified at all but finitely many prime ideals of $\mathcal{O}_E$ in the sense that the inertia group at each of these prime ideals acts trivially on V.
(ii) V is de Rham at each prime ideal $\mathfrak{p}$ of $\mathcal{O}_E$ lying over p in the sense that the restriction of V to $\Gamma_{E_{\mathfrak{p}}}$ is de Rham.

Remark. Let us explain the necessity of properties (i) and (ii). We take $E = \mathbb{Q}$ for notatoinal simplicity. It is a standard fact that every proper smooth variety X over $\mathbb{Q}$ has good reduction at all but finitely many primes. If X has good reduction at a prime $\ell \neq p$, we deduce from a general fact about the étale cohomology that there exists a $\Gamma_{\mathbb{Q}_\ell}$-equivariant isomorphism

$$H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_\ell}, \mathbb{Q}_p) \cong H^n_{\text{ét}}(\overline{X}_{\overline{\mathbb{F}}_\ell}, \mathbb{Q}_p),$$

where $\overline{X}$ denotes the mod ℓ reduction of X, and thus find that the inertia group at ℓ acts trivially on the Tate twists of $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_\ell}, \mathbb{Q}_p)$ and their subquotients. Moreover, Theorem 1.2.3 in Chapter I shows that $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ is de Rham, which in turn implies that the Tate twists of $H^n_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ and their subquotients are de Rham by Lemma 2.4.6 and Proposition 2.4.10.

Conjecture 2.4.19 has a very surprising implication that the behavior of a p-adic Γ_E-representation V at prime ideals lying over p affects the behavior of V at other prime ideals. We continue to take $E = \mathbb{Q}$ for notational simplicity. If the p-adic étale cohomology of a proper smooth variety X over $\mathbb{Q}$ is unramified at a prime ℓ, the eigenvalues of the Frobenius element in $\Gamma_{\mathbb{Q}_\ell}$ turn out to be algebraic numbers. Hence for a p-adic $\Gamma_{\mathbb{Q}}$-representation V which are unramified at almost all primes, being de Rham at p should force the eigenvalues of the Frobenius at all unramified primes to be algebraic.

If V is one-dimensional, Conjecture 2.4.19 holds essentially by results of Weil [**Wei56**] and Serre [**Ser68**]. For $E = \mathbb{Q}$, the key fact is that every one-dimensional p-adic $\Gamma_{\mathbb{Q}}$-representation with properties (i) and (ii) corresponds to a Tate twist of a continuous character $\eta : \Gamma_{\mathbb{Q}} \to \mathbb{Q}_p^\times$ with finite order. We may regard the values of such a character η as elements of a number field which contains sufficiently many roots of unity and thus deduce that η arises from the etale cohomology of a zero-dimensional smooth variety over $\mathbb{Q}$. For a general number field E, a similar argument applies after some modifications.

If V is two-dimensional, the results of Kisin [**Kis09**], Emerton [**Eme11**], and Pan [**Pan22**] verify Conjecture 2.4.19 under some additional assumptions. These results exploit a tidy connection between two-dimensional Galois representations and certain holomorphic complex functions called *modular forms*. A key ingredient for these results is a refinement of the method developed by Taylor-Wiles [**TW95**], commonly referred to as the *Taylor-Wiles patching*, which has numerous applications including the proof of Fermat's Last Theorem by Wiles [**Wil95**].

The natural local analogue of Conjecture 2.4.19 is false. In other words, if K is a finite extension of $\mathbb{Q}_p$ there exists a de Rham Γ_K-representation which is not a subquotient of $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p(m))$ for a proper smooth variety X over K. The main issue is that the p-adic étale cohomology of a proper smooth variety X over K satisfies certain arithmetic property; for example, if X has good reduction the eigenvalues for the Frobenius elements in Γ_K must be algebraic.

3. Crystalline representations

In this section, we define and study the crystalline period ring and crystalline representations. Our primary references for this section are the notes of Brinon-Conrad [**BC**, §9] and the notes of Fontaine-Ouyang [**FO**, §7].

3.1. The crystalline period ring B_{cris}

Throughout this section, we denote by K_0 the fraction field of $W(k)$. Let us recall that we have fixed a cyclotomic uniformizer $t = \log(\varepsilon)$ of B_{dR}^+ for some $\varepsilon \in \mathbb{Z}_p(1)$ and a distinguished element $\xi = [p^\flat] - p \in A_{\text{inf}}$ for some $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$.

Definition 3.1.1. The *integral crystalline period ring*, denoted by A_{cris}, is the p-adic completion of the A_{inf}-subalgebra A_{cris}^0 in $A_{\text{inf}}[1/p]$ generated by the elements $\xi^n/n!$ with $n \geq 0$.

LEMMA 3.1.2. The elements $\xi^n/n! \in A_{\text{cris}}^0$ with $n \geq 0$ generate A_{cris}^0 as an A_{inf}-module.

PROOF. Since we have

$$\frac{\xi^m}{m!} \cdot \frac{\xi^n}{n!} = \binom{m+n}{n} \frac{\xi^{m+n}}{(m+n)!} \quad \text{for } m, n \geq 0,$$

the assertion is straightforward to verify. □

PROPOSITION 3.1.3. The ring $A_{\text{inf}}[[\xi/p]]$ is a p-adically complete subring of B_{dR}^+.

PROOF. Proposition 2.2.17 shows that $A_{\text{inf}}[[\xi/p]]$ is a subring of B_{dR}^+. Let us consider the natural ring homomorphism

$$\eta : A_{\text{inf}}[[\xi/p]] \longrightarrow \varprojlim_n A_{\text{inf}}[[\xi/p]]/p^n A_{\text{inf}}[[\xi/p]].$$

We wish to show that η is an isomorphism.

Take an arbitrary element $b \in \ker(\eta)$. For every $n \geq 1$, we may write

$$b = p^n \sum_{i=0}^{\infty} a_{n,i} \frac{\xi^i}{p^i} \quad \text{with } a_{n,i} \in A_{\text{inf}}.$$

We have $\theta_{\text{dR}}^+(b) = p^n \theta_{\text{dR}}^+(a_{n,0}) \in p^n \mathcal{O}_{\mathbb{C}_K}$ for each $n \geq 1$ and thus find $\theta_{\text{dR}}^+(b) = 0$. We see that each $a_{n,0}$ satisfies the equality $\theta_{\text{dR}}^+(a_{n,0}) = 0$, which means by Proposition 2.2.12 that each $a_{n,0}$ is divisible by ξ in A_{inf}. Moreover, we obtain the relation

$$\frac{b}{\xi} = p^{n-1} \left((\frac{p a_{n,0}}{\xi} + a_{n,1}) + \sum_{i=2}^{\infty} a_{n,i} \frac{\xi^{i-1}}{p^{i-1}} \right) \in p^{n-1} A_{\text{inf}}[[\xi/p]] \quad \text{for every } n \geq 1$$

and in turn find $b/\xi \in \ker(\eta)$. Now a simple induction shows that b is infinitely divisible by ξ in $A_{\text{inf}}[[\xi/p]]$ and thus is zero. We deduce that η is injective.

It remains to show that η is surjective. Choose an arbitrary sequence (b'_n) in $A_{\text{inf}}[[\xi/p]]$ with $b'_{n+1} - b'_n \in p^n A_{\text{inf}}[[\xi/p]]$ for every $n \geq 0$. For each $n \geq 0$, we may write

$$b'_{n+1} - b'_n = p^n \sum_{i=0}^{\infty} a'_{n,i} \frac{\xi^i}{p^i} \quad \text{with } a'_{n,i} \in A_{\text{inf}}.$$

We take $a'_i := \sum_{n=0}^{\infty} a'_{n,i} p^n \in A_{\text{inf}}$ and see that $b' := b'_0 + \sum_{i=0}^{\infty} a'_i \frac{\xi^i}{p^i} \in A_{\text{inf}}[[\xi/p]]$ satisfies the relation $b' - b'_n \in p^n A_{\text{inf}}[[\xi/p]]$ for every $n \geq 0$. Hence η is surjective as desired. □

Proposition 3.1.4. The ring A_{cris} is naturally a subring of B_{dR}^+ with an identification

$$A_{\mathrm{cris}} = \left\{ \sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!} \in B_{\mathrm{dR}}^+ : a_n \in A_{\mathrm{inf}} \text{ with } \lim_{n\to\infty} a_n = 0 \right\}.$$

Proof. Since A_{cris}^0 is naturally a subring of $A_{\mathrm{inf}}[[\xi/p]]$ by construction, Proposition 3.1.3 yields canonical injective maps

$$A_{\mathrm{cris}}^0 \hookrightarrow A_{\mathrm{cris}} \hookrightarrow A_{\mathrm{inf}}[[\xi/p]] \hookrightarrow B_{\mathrm{dR}}^+.$$

In addition, it is not difficult to see that every element $b = \sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!} \in B_{\mathrm{dR}}^+$ with $a_n \in A_{\mathrm{inf}}$ and $\lim_{n\to\infty} a_n = 0$ lies in A_{cris}; indeed, we take the maximum integer n_m with $a_{n_m} \notin p^m A_{\mathrm{inf}}$ for each $m \geq 1$ and set $b_m := \sum_{n=0}^{n_m} a_n \frac{\xi^n}{n!} \in A_{\mathrm{cris}}^0$ to find $b - b_m \in p^m A_{\mathrm{inf}}[[\xi/p]]$. Let us now consider an arbitrary sequence (b_n') in A_{cris}^0 with $b_{n+1}' - b_n' \in p^n A_{\mathrm{cris}}^0$ for every $n \geq 0$. We note that each $b_{n+1}' - b_n'$ admits an expression

$$b_{n+1}' - b_n' = p^n \sum_{i=0}^{\infty} a_{n,i}' \frac{\xi^i}{i!} \quad \text{with } a_{n,i}' \in A_{\mathrm{inf}}$$

where the sum has finitely many nonzero terms. Take $a_i' := \sum_{n=0}^{\infty} a_{n,i}' p^n \in A_{\mathrm{inf}}$ for each $i \geq 0$ and set $b' := b_0' + \sum_{i=0}^{\infty} a_i' \frac{\xi^i}{i!} \in B_{\mathrm{dR}}^+$. We find $b' - b_n' \in p^n A_{\mathrm{inf}}[[\xi/p]]$ for every $n \geq 0$ and in turn see that (b_n') converges to b'. Moreover, we have $\lim_{i\to\infty} a_i' = 0$ as there exists an increasing sequence (l_n) in $\mathbb{Z}$ with $a_{n,i}' = 0$ for each $i > l_n$. Hence we establish the desired assertion. ☐

Proposition 3.1.5. The element $t \in B_{\mathrm{dR}}^+$ lies in A_{cris}.

Proof. Since we have $[\varepsilon] - 1 = \xi c$ for some $c \in A_{\mathrm{inf}}$ by Lemma 2.2.22, we find

$$t = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} (n-1)! c^n \cdot \frac{\xi^n}{n!}.$$

Now we observe the identity $\lim_{n\to\infty} (n-1)! c^n = 0$ and consequently deduce the desired assertion from Proposition 3.1.4. ☐

Definition 3.1.6. The *crystalline period ring* is $B_{\mathrm{cris}} := B_{\mathrm{cris}}^+[1/t]$ with $B_{\mathrm{cris}}^+ := A_{\mathrm{cris}}[1/p]$.

Remark. Let us explain Fontaine's insight behind the construction of B_{cris}. As briefly discussed in Chapter I, Fontaine introduced B_{cris} to formulate the crystalline comparison isomorphism. Given a proper smooth K-variety X with a smooth reduction $\overline{X}$ over k, the crystalline cohomology $H_{\mathrm{cris}}^n(\overline{X}, W(k))$ admits a natural Frobenius-semilinear endomorphism and a canonical K-linear isomorphism $H_{\mathrm{cris}}^n(\overline{X}, W(k))[1/p] \otimes_{K_0} K \cong H_{\mathrm{dR}}^n(X/K)$. Fontaine sought the ring B_{cris} as a $(\mathbb{Q}_p, \Gamma_K)$-regular subring of B_{dR} with a natural extension of the Frobenius automorphism φ_{inf} on $A_{\mathrm{inf}}[1/p] = W(\mathcal{O}_F)[1/p]$. The ring B_{dR} does not admit a natural extension of φ_{inf} since $\ker(\theta[1/p])$ is not stable under φ_{inf}. Fontaine discovered that A_{cris}^0 is stable under φ_{inf} and consequently showed that φ_{inf} canonically extends to an endomorphism of A_{cris}. The only issue with A_{cris} is that it is not $(\mathbb{Q}_p, \Gamma_K)$-regular, which Fontaine resolved by taking the ring $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p, 1/t]$.

PROPOSITION 3.1.7. The ring B_{cris} admits an identification $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$.

PROOF. Since we have $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p, 1/t]$, we wish to show that p is a unit in $A_{\mathrm{cris}}[1/t]$. It suffices to prove the relation $t^{p-1} \in pA_{\mathrm{cris}}$. Let us set

$$\check{t} := \sum_{n=1}^{p} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} \in B_{\mathrm{dR}}^+$$

We may write $[\varepsilon] - 1 = \xi a$ for some $a \in A_{\mathrm{inf}}$ by Lemma 2.2.22 and in turn find

$$t - \check{t} = \sum_{n=p+1}^{\infty} (-1)^{n+1} \frac{([\varepsilon]-1)^n}{n} = \sum_{n=p+1}^{\infty} (-1)^{n+1}(n-1)! a^n \cdot \frac{\xi^n}{n!}.$$

Since $(n-1)!$ is divisible by p for every $n > p$, we have $t - \check{t} \in pA_{\mathrm{cris}}$ by Proposition 3.1.4. We wish to prove the relation $\check{t}^{p-1} \in pA_{\mathrm{cris}}$. In the definition of $\check{t}$, the terms with $n < p$ are all divisible by $[\varepsilon] - 1$ in A_{cris}; in other words, we may write

$$\check{t} = ([\varepsilon]-1)\left(b + (-1)^{p+1}\frac{([\varepsilon]-1)^{p-1}}{p}\right)$$

for some $b \in A_{\mathrm{cris}}$. Hence it suffices to establish the relation $([\varepsilon]-1)^{p-1} \in pA_{\mathrm{cris}}$. Since we have $([\varepsilon]-1) - [\varepsilon - 1] \in pA_{\mathrm{inf}} \subseteq pA_{\mathrm{cris}}$, it is enough to prove the relation $[(\varepsilon-1)^{p-1}] \in pA_{\mathrm{cris}}$. We apply Lemma 2.2.22 to find

$$\nu^\flat\left((\varepsilon-1)^{p-1}\right) = p = \nu^\flat\left((p^\flat)^p\right)$$

and in turn deduce that $[(\varepsilon-1)^{p-1}]$ is divisible by $[p^\flat]^p = (\xi + p)^p$. Now we obtain the desired relation by observing that $\xi^p = p \cdot (p-1)! \cdot (\xi^p/p!)$ is divisible by p in A_{cris}. □

PROPOSITION 3.1.8. The ring B_{cris} is naturally a filtered K_0-subalgebra of B_{dR} which is stable under the action of Γ_K.

PROOF. Proposition 3.1.4 yields the relation

$$A_{\mathrm{inf}}[1/p] \subseteq A_{\mathrm{cris}}[1/p] = B_{\mathrm{cris}}^+ \subseteq B_{\mathrm{cris}} \subseteq B_{\mathrm{dR}}.$$

In addition, Proposition 2.2.19 shows that the natural homomorphism $K \to B_{\mathrm{dR}}$ extends the canonical homomorphism $K_0 \to A_{\mathrm{inf}}[1/p]$. Hence B_{cris} is naturally a filtered K_0-subalgebra of B_{dR} with $\mathrm{Fil}^n(B_{\mathrm{cris}}) = B_{\mathrm{cris}} \cap t^n B_{\mathrm{dR}}^+$ for each $n \in \mathbb{Z}$.

It remains to show that B_{cris} is stable under the action of Γ_K. Let us work with the identification $B_{\mathrm{cris}} = A_{\mathrm{cris}}[1/t]$ given by Proposition 3.1.7. Since Γ_K acts on t via χ as noted in Theorem 2.2.26, we only need to prove that A_{cris} is stable under the action of Γ_K. Take an arbitrary element $\gamma \in \Gamma_K$ and an arbitrary sequence (a_n) in A_{inf} with $\lim_{n\to\infty} a_n = 0$. We observe that $\ker(\theta)$ is stable under the Γ_K-action by Theorem 2.2.26 and in turn find $\gamma(\xi) = b_\gamma \xi$ for some $b_\gamma \in A_{\mathrm{inf}}$ by Proposition 2.2.12. In addition, we note that the Γ_K-action is continuous on B_{dR}^+ with respect to the discrete valuation topology and on A_{inf} with respect to the p-adic topology. Now we apply Proposition 3.1.4 to obtain the relation

$$\gamma\left(\sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!}\right) = \sum_{n=0}^{\infty} \gamma(a_n) b_\gamma^n \frac{\xi^n}{n!} \in A_{\mathrm{cris}}$$

and consequently deduce the desired assertion. □

Remark. It is worthwhile to mention that $\mathrm{Fil}^0(B_{\mathrm{cris}}) = B_{\mathrm{cris}} \cap B_{\mathrm{dR}}^+$ is not equal to B_{cris}^+. In fact, we can show that $\dfrac{[\varepsilon^{1/p^2}]-1}{[\varepsilon^{1/p}]-1}$ lies in $B_{\mathrm{cris}} \cap B_{\mathrm{dR}}^+$ but not in B_{cris}^+.

In order to study the Γ_K-action and the filtration on B_{cris}, we invoke the following crucial result without a proof.

PROPOSITION 3.1.9. The natural Γ_K-equivariant map $B_{\text{cris}} \otimes_{K_0} K \to B_{\text{dR}}$ is injective.

Remark. The assertion is evident if we have $K = K_0$. However, the proof for the general case is surprisingly difficult. Moreover, the original proof by Fontaine [**Fon94a**] is incomplete. We refer curious readers to the article of Colmez [**Col02**, Proposition 8.12] for a complete proof, which involves an enlargement $B_{\max}$ of B_{cris}. The key point is that we can construct B_{dR} with $A_{\inf}[1/p]_K := A_{\inf}[1/p] \otimes_{K_0} K$ in place of $A_{\inf}[1/p]$; indeed, the K-algebra homomorphism $\theta[1/p]_K : A_{\inf}[1/p]_K \to \mathbb{C}_K$ induced by $\theta[1/p]$ turns out to yield a natural isomorphism

$$B_{\text{dR}}^+ \cong \varprojlim_i A_{\inf}[1/p]_K / \ker(\theta[1/p]_K)^i.$$

PROPOSITION 3.1.10. There exists a natural Γ_K-equivariant graded K-algebra isomorphism

$$\text{gr}(B_{\text{cris}} \otimes_{K_0} K) \cong B_{\text{HT}}.$$

PROOF. Theorem 2.2.26 and Proposition 3.1.9 show that the natural filtered K-algebra homomorphism $B_{\text{cris}} \otimes_{K_0} K \to B_{\text{dR}}$ yields an injective graded K-algebra homomorphism

$$\text{gr}(B_{\text{cris}} \otimes_{K_0} K) \hookrightarrow \text{gr}(B_{\text{dR}}) \cong B_{\text{HT}}. \tag{3.1}$$

Since each $\text{Fil}^n(B_{\text{cris}}) = B_{\text{cris}} \cap t^n B_{\text{dR}}^+$ is stable under the Γ_K-action by Theorem 2.2.26, we obtain a natural action of Γ_K on $\text{gr}(B_{\text{cris}} \otimes_{K_0} K)$ and in turn deduce that the map (3.1) is Γ_K-equivariant. Meanwhile, the map (3.1) gives rise to an injective K-algebra homomorphism

$$\text{gr}^0(B_{\text{cris}} \otimes_{K_0} K) \hookrightarrow \text{gr}^0(B_{\text{dR}}) \cong \mathbb{C}_K,$$

which is indeed an isomorphism as the image of $B_{\text{cris}} \otimes_{K_0} K$ in B_{dR} contains $A_{\inf}[1/p]$ and consequently maps onto $\mathbb{C}_K \cong B_{\text{dR}}^+ / \ker(\theta_{\text{dR}}^+)$ by Proposition 2.2.6. Hence the map (3.1) is a graded $\mathbb{C}_K$-algebra homomorphism. Moreover, each $\text{gr}^n(B_{\text{cris}} \otimes_{K_0} K)$ contains a nonzero element $t^n \otimes 1$ while each $\text{gr}^n(B_{\text{dR}})$ has dimension 1 over $\mathbb{C}_K$. We deduce that the injective map (3.1) is an isomorphism, thereby completing the proof. □

LEMMA 3.1.11. Let L be an algebraic extension of K and $\widehat{L}$ be its p-adic completion.

(1) The residue fields of L and $\widehat{L}$ are naturally isomorphic.
(2) The absolute Galois groups of L and $\widehat{L}$ are naturally isomorphic.

PROOF. Statement (1) follows immediately from a standard fact about completions stated in the Stacks project [**Sta**, Tag 05GG]. Hence we only need to establish statement (2). The p-adic completion $\mathbb{C}_K$ of $\overline{K} = \overline{L}$ evidnetly contains $\widehat{L}$. Since $\mathbb{C}_K$ is algebraically closed by Proposition 3.1.10 in Chapter II, it also contains $\overline{\widehat{L}}$ and its p-adic completion $\mathbb{C}_{\widehat{L}}$. Meanwhile, we observe that every element of $\mathbb{C}_K$ is an element of $\mathbb{C}_{\widehat{L}}$ as K lies in $\widehat{L}$. Therefore we find $\mathbb{C}_K = \mathbb{C}_{\widehat{L}}$ and in turn obtain a natural identification

$$\Gamma_{\widehat{L}} = \text{Aut}_{\widehat{L}}(\overline{\widehat{L}}) \cong \text{Aut}_{\widehat{L}}(\mathbb{C}_{\widehat{L}}) = \text{Aut}_{\widehat{L}}(\mathbb{C}_K) \cong \text{Aut}_L(\mathbb{C}_K) \cong \text{Aut}_L(\overline{K}) = \Gamma_L$$

where we take all automorphisms to be continuous. □

Example 3.1.12. The *completed unramified closure* of K, denoted by $\widehat{K^{\text{un}}}$, is the p-adic completion of the maximal unramified extension K^{un} of K. Lemma 3.1.11 shows that $\widehat{K^{\text{un}}}$ is a p-adic field with residue field $\overline{k}$ and absolute Galois group I_K.

Remark. We can naturally identify $\widehat{K^{\text{un}}}$ with $K \otimes_{K_0} W(\overline{k})[1/p]$.

Proposition 3.1.13. An element $b \in (B_{\mathrm{dR}}^+)^\times$ is algebraic over the fraction field $\widehat{K_0^{\mathrm{un}}}$ of $W(\overline{k})$ if Γ_K acts on b via a character $\eta : \Gamma_K \to \mathbb{Q}_p^\times$.

Proof. Theorem 2.2.26 implies that Γ_K acts on $\hat{b} := \theta_{\mathrm{dR}}^+(b) \in \mathbb{C}_K$ via the character η. We see that η is continuous as the Γ_K-action on $\mathbb{C}_K$ is continuous. In addition, we may regard $\hat{b} = \theta_{\mathrm{dR}}^+(b)$ as an element of $\mathbb{C}_K(\eta^{-1})^{\Gamma_K}$ which is nonzero for being the image of a unit in B_{dR}^+. Therefore Theorem 1.1.7 shows that $\eta(I_K)$ is finite.

Since $\widehat{K^{\mathrm{un}}}$ is a p-adic field with residue field $\overline{k}$ and absolute Galois group I_K as noted in Example 3.1.12, it has a finite extension L such that η is trivial on Γ_L. Lemma 2.4.16 implies that L is a p-adic field with residue field $\overline{k}$. Now we apply Proposition 2.2.19 to deduce that L is finite over the fraction field $\widehat{K_0^{\mathrm{un}}}$ of $W(\overline{k})$ and fits into a natural commutative diagram

$$\begin{array}{ccc} \widehat{K_0^{\mathrm{un}}} & \hookrightarrow & A_{\mathrm{inf}}[1/p] \\ \downarrow & & \downarrow \\ L & \hookrightarrow & B_{\mathrm{dR}}^+ \\ & \searrow & \downarrow {\scriptstyle \theta_{\mathrm{dR}}^+} \\ & & \mathbb{C}_K \end{array} \tag{3.2}$$

with all maps being Γ_K-equivariant. We identify L as a subfield of B_{dR} via the diagram (3.2).

It suffices to show that b lies in L. Suppose for contradiction that b does not belong to L. Since Γ_K acts on $\hat{b}$ via η, we have $\hat{b} \in \mathbb{C}_K^{\Gamma_L} = \mathbb{C}_L^{\Gamma_L} = L$ by Theorem 3.1.11 in Chapter II and thus find $b \neq \hat{b}$. Moreover, the diagram (3.2) yields the identity $\theta_{\mathrm{dR}}^+(\hat{b}) = \hat{b} = \theta_{\mathrm{dR}}^+(b)$, which implies that there exists a unique integer $m > 0$ with $b - \hat{b} \in t^m B_{\mathrm{dR}}^+$ and $b - \hat{b} \notin t^{m+1} B_{\mathrm{dR}}^+$. We see that Γ_K acts via η on the nonzero image of $b - \hat{b}$ in $\mathbb{C}_K(m) \cong t^m B_{\mathrm{dR}}^+ / t^{m+1} B_{\mathrm{dR}}^+$ given by Theorem 2.2.26. Hence we deduce from Theorem 1.1.7 that I_K has a finite image under $\chi^m = \eta \cdot (\eta^{-1} \chi^m)$, thereby obtaining a desired contradiction by Lemma 1.1.8. $\square$

Theorem 3.1.14 (Fontaine [**Fon94a**]). The ring B_{cris} is $(\mathbb{Q}_p, \Gamma_K)$-regular with $B_{\mathrm{cris}}^{\Gamma_K} \cong K_0$.

Proof. The ring B_{cris} is a subring of the field B_{dR} and thus is an integral domain. Proposition 3.1.8 implies that the fraction field C_{cris} of B_{cris} is a K_0-subalgebra of B_{dR} which is stable under the action of Γ_K. In addition, Proposition 3.1.9 and Theorem 2.2.26 together yield natural injective K-algebra homomorphisms

$$B_{\mathrm{cris}}^{\Gamma_K} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}^{\Gamma_K} \cong K \quad \text{and} \quad C_{\mathrm{cris}}^{\Gamma_K} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}^{\Gamma_K} \cong K.$$

Therefore we have $K_0 \cong B_{\mathrm{cris}}^{\Gamma_K} \cong C_{\mathrm{cris}}^{\Gamma_K}$.

It remains to prove that every nonzero $b \in B_{\mathrm{cris}}$ with $\mathbb{Q}_p b$ being stable under the Γ_K-action is a unit. We apply Proposition 2.2.23 to write $b = t^n b'$ for some $b' \in (B_{\mathrm{dR}}^+)^\times$ and $n \in \mathbb{Z}$. We observe that t is a unit in $B_{\mathrm{cris}} = B_{\mathrm{cris}}^+[1/t]$ and in turn find $b' = bt^{-n} \in B_{\mathrm{cris}}$. Moreover, Theorem 2.2.26 implies that $\mathbb{Q}_p b'$ is stable under the Γ_K-action. Hence we may replace b by b' to assume that b is a unit in B_{dR}^+. Proposition 3.1.13 yields a polynomial equation

$$b^d + c_1 b^{d-1} + \cdots + c_{d-1} b + c_d = 0 \quad \text{with } c_d \neq 0$$

where each c_i is an element in the fraction field $\widehat{K_0^{\mathrm{un}}}$ of $W(\overline{k})$. Now we find

$$b^{-1} = -c_d^{-1}(b^{d-1} + c_1 b^{d-2} + \cdots + c_{d-1}) \in B_{\mathrm{cris}}$$

by noting that $\widehat{K_0^{\mathrm{un}}}$ naturally embeds into B_{cris}, thereby completing the proof. $\square$

Our final objective in this subsection is to construct the Frobenius endomorphism on the crystalline period ring B_{cris}.

Lemma 3.1.15. The Frobenius automorphism of A_{inf} uniquely extends to a Γ_K-equivariant endomorphism φ^+ on B_{cris}^+ which is continuous with respect to the p-adic topology.

Proof. The Frobenius automorphism of $A_{\text{inf}} = W(\mathcal{O}_F)$ uniquely extends to an automorphism on $A_{\text{inf}}[1/p]$, which we denote by φ_{inf}. Since we have an equality

$$\varphi_{\text{inf}}(\xi) = [(p^\flat)^p] - p = [p^\flat]^p - p = (\xi + p)^p - p, \tag{3.3}$$

we may write $\varphi_{\text{inf}}(\xi) = \xi^p + pa$ for some $a \in A_{\text{inf}}$. We find

$$\varphi_{\text{inf}}(\xi) = p \cdot (a + (p-1)! \cdot (\xi^p/p!))$$

and in turn obtain the relation

$$\varphi_{\text{inf}}(\xi^n/n!) = (p^n/n!) \cdot (a + (p-1)! \cdot (\xi^p/p!))^n \in A_{\text{cris}}^0 \quad \text{for each } n \geq 1$$

by observing that $p^n/n!$ is an element of $\mathbb{Z}_p$. Hence A_{cris}^0 is stable under φ_{inf}. Moreover, the automorphism φ_{inf} on $A_{\text{inf}}[1/p]$ is by construction Γ_K-equivariant and continuous with respect to the p-adic topology. Now we note that the Γ_K-action on A_{cris} is continuous with respect to the p-adic topology and in turn establish the desired assertion by the identity $B_{\text{cris}}^+ = A_{\text{cris}}[1/p]$. □

Remark. The equality (3.3) shows that $\varphi_{\text{inf}}(\xi)$ is not divisible by ξ, which implies that $\ker(\theta)$ is not stable under φ_{inf}. Hence the endomorphism φ^+ on B_{cris}^+ (or the Frobenius endomorphism on B_{cris} that we are about to construct) is not filtered.

Proposition 3.1.16. The Frobenius automorphism of A_{inf} canonically extends to a Γ_K-equivariant endomorphism φ on B_{cris} with $\varphi(t) = pt$.

Proof. By Lemma 3.1.15, the Frobenius automorphism of A_{inf} uniquely extends to a Γ_K-equivariant continuous endomorphism φ^+ on B_{cris}^+. Hence we apply Proposition 2.2.23 to obtain the equality

$$\varphi^+(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\varphi([\varepsilon]) - 1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon^p] - 1)^n}{n} = \log(\varepsilon^p) = p \log(\varepsilon) = pt.$$

Since Γ_K acts on t via χ as noted in Theorem 2.2.26, we deduce that φ^+ uniquely extends to a Γ_K-equivariant endomorphism φ on $B_{\text{cris}} = B_{\text{cris}}^+[1/t]$. □

Remark. The endomorphism φ is not continuous on B_{cris}, even though it is a unique extension of the continuous endomorphism φ^+ on B_{cris}^+. The issue is that, as pointed out by Colmez [**Col98**], the natural topology on B_{cris}^+ induced by the p-adic topology on A_{cris} does not agree with the subspace topology inherited from B_{cris}. In fact, it is not hard to show that the sequence $\left(\dfrac{\xi^{p^n-1}}{(p^n-1)!}\right)$ converges to 0 in B_{cris} but does not converge to 0 in B_{cris}^+.

Definition 3.1.17. We refer to the map φ in Proposition 3.1.16 as the *Frobenius endomorphism* of B_{cris} and write $B_e := B_{\text{cris}}^{\varphi=1}$.

3.2. Properties of crystalline representations

For the rest of this section, we denote by σ the Frobenius automorphism of K_0 and by Vect_{K_0} the category of K_0-vector spaces. Let us invoke the following result without a proof.

PROPOSITION 3.2.1. The Frobenius endomorphism of B_{cris} is injective.

Remark. We will present a proof of Proposition 3.2.1 in Chapter IV. We refer curious readers to the article of Brinon [**Bri22**, Theorem 1] for another proof which does not involve materials covered in Chapter IV.

Definition 3.2.2. For a K_0-vector space V, we write $V_K = V \otimes_{K_0} K$.

(1) A *filtered isocrystal* over K is an isocrystal D over K_0 such that D_K is a filtered K-vector space.
(2) Given filtered isocrystals D and D' over K, a morphism $f : D \to D'$ of isocrystals over K_0 is *K-filtered* if the induced K-linear map $f_K : D_K \to D'_K$ is filtered.

Remark. A K-filtered isomorphism of isocrystals is a bijective K-filtered morphism of isocrystals with a K-filtered inverse.

Example 3.2.3. Every proper smooth K-variety X with a smooth reduction $\overline{X}$ over k yields a filtered isocrystal $H^n_{\mathrm{cris}}(\overline{X}/K_0)$ over K with $H^n_{\mathrm{cris}}(\overline{X}/K_0)_K \cong H^n_{\mathrm{dR}}(X/K)$.

PROPOSITION 3.2.4. Let D be a filtered isocrystal over K.

(1) Given a filtered isocrystal D' over K, the tensor product $D \otimes_{K_0} D'$ is naturally a filtered isocrystal over K.
(2) The dual $D^\vee = \mathrm{Hom}_{K_0}(D, K_0)$ is naturally a filtered isocrystal over K.

PROOF. The assertions follow from Lemma 2.3.17 in Chapter II and Proposition 2.3.6. □

LEMMA 3.2.5. Given a finite dimensional vector space D over K_0, every injective σ-semilinear map $f : D \to D$ is bijective.

PROOF. The additivity of f implies that $f(D)$ is closed under addition. In fact, $f(D)$ is a subspace of D over K_0, as for arbitrary $c \in K_0$ and $v \in D$ we have

$$cf(v) = \sigma(\sigma^{-1}(c))f(v) = f(\sigma^{-1}(c)v) \in f(D).$$

Let us choose a K_0-basis (e_i) for D. Since f is σ-semilinear and injective, every relation $\sum c_i f(e_i) = 0$ with $c_i \in K_0$ yields a relation $\sum \sigma(c_i)e_i = 0$. We deduce that the vectors $f(e_i)$ form a K_0-basis for D and consequently find $f(D) = D$. □

PROPOSITION 3.2.6. Every p-adic Γ_K-representation V naturally yields a filtered isocrystal $D_{\mathrm{cris}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K}$ over K with Frobenius automorphism $1 \otimes \varphi$ and

$$\mathrm{Fil}^n(D_{\mathrm{cris}}(V)_K) = (V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^n(B_{\mathrm{cris}} \otimes_{K_0} K))^{\Gamma_K} \quad \text{for each } n \in \mathbb{Z}. \tag{3.4}$$

PROOF. Theorem 1.2.1 and Theorem 3.1.14 together imply that $D_{\mathrm{cris}}(V)$ is a finite dimensional K_0-vector space. In addition, we find

$$D_{\mathrm{cris}}(V)_K = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K} \otimes_{K_0} K = (V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K))^{\Gamma_K}$$

and in turn deduce from Proposition 3.1.9 that $D_{\mathrm{cris}}(V)_K$ is a filtered K-vector space with the identification (3.4). Meanwhile, since $1 \otimes \varphi$ is σ-semilinear by the fact that φ extends σ, it is bijective on $D_{\mathrm{cris}}(V)$ by Proposition 3.2.1 and Lemma 3.2.5. Therefore we establish the desired assertion. □

Definition 3.2.7. Let V be a p-adic Γ_K-representation.

(1) We refer to $D_{\text{cris}}(V)$ in Proposition 3.2.6 as the *filtered isocrystal associated to V*.
(2) We say that V is *crystalline* if it is B_{cris}-admissible.

Example 3.2.8. Let us present two essential examples of crystalline Γ_K-representations.

(1) For every proper smooth K-variety X with a smooth reduction $\overline{X}$ over k, the étale cohomology $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline with a natural isomorphism
$$D_{\text{cris}}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \cong H^n_{\text{cris}}(\overline{X}/K_0)$$
as briefly discussed in Chapter I.
(2) For every p-divisible group G over $\mathcal{O}_K$ with special fiber $\overline{G} = G \times_{\mathcal{O}_K} k$, the rational Tate module $V_p(G)$ is crystalline with a natural isomorphism
$$D_{\text{cris}}(V_p(G)) \cong \mathbb{D}(\overline{G})[1/p]$$
as proved by Fontaine [**Fon82**, §6].

Proposition 3.2.9. If a p-adic Γ_K-representation V is crystalline, it is de Rham with a natural K-linear filtered isomorphism
$$D_{\text{cris}}(V)_K \cong D_{\text{dR}}(V).$$

Proof. Proposition 3.1.8 and Proposition 3.1.9 together show that $B_{\text{cris}} \otimes_{K_0} K$ is naturally a filtered K-subalgebra of B_{dR} with
$$\text{Fil}^n(B_{\text{cris}} \otimes_{K_0} K) = (B_{\text{cris}} \otimes_{K_0} K) \cap \text{Fil}^n(B_{\text{dR}}) \quad \text{for every } n \in \mathbb{Z}.$$
Therefore Proposition 3.2.6 yields a natural injective K-linear filtered map
$$D_{\text{cris}}(V)_K = (V \otimes_{\mathbb{Q}_p} (B_{\text{cris}} \otimes_{K_0} K))^{\Gamma_K} \longrightarrow (V \otimes_{\mathbb{Q}_p} B_{\text{dR}})^{\Gamma_K} = D_{\text{dR}}(V)$$
with an identification
$$\text{Fil}^n(D_{\text{cris}}(V)_K) = D_{\text{cris}}(V)_K \cap \text{Fil}^n(D_{\text{dR}}(V)) \quad \text{for every } n \in \mathbb{Z}.$$
In addition, we find
$$\dim_{K_0} D_{\text{cris}}(V) = \dim_K D_{\text{cris}}(V)_K \leq \dim_K D_{\text{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$$
where the last inequality follows from Theorem 1.2.1. Since V is crystalline, we see that both inequalities should be equalities and in turn establish the desired assertion. □

Example 3.2.10. Every Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ is crystalline; indeed, the inequality
$$\dim_K D_{\text{cris}}(\mathbb{Q}_p(n)) \leq \dim_{\mathbb{Q}_p} \mathbb{Q}_p(n) = 1$$
given by Theorem 1.2.1 is an equality, as $D_{\text{cris}}(\mathbb{Q}_p(n)) = (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_K}$ contains a nonzero element $1 \otimes t^{-n}$ by Theorem 2.2.26. Moreover, $D_{\text{cris}}(\mathbb{Q}_p(n))$ is naturally isomorphic to the simple isocrystal of slope $-n$ with identifications
$$\text{Fil}^m(D_{\text{cris}}(\mathbb{Q}_p(n))_K) \cong \begin{cases} K & \text{for } m \leq -n, \\ 0 & \text{for } m > -n \end{cases}$$
given by Example 2.4.5 and Proposition 3.2.9.

Lemma 3.2.11. Given an integer n, a p-adic Γ_K-representation V is crystalline if and only if its Tate twist $V(n)$ is crystalline.

Proof. Since we have $V(n) \cong V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$ and $V \cong V(n) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-n)$, the assertion follows from Proposition 1.2.4 and Example 3.2.10. □

For the rest of this section, we denote by $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ the category of crystalline Γ_K-representations and by MF_K^{φ} the category of filtered isocrystals over K.

Proposition 3.2.12. Every $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ admits a natural Γ_K-equivariant isomorphism

$$D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$$

which is compatible with the Frobenius endomorphisms and induces a filtered isomorphism

$$D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K).$$

Proof. Since V is crystalline, Theorem 1.2.1 implies that the natural B_{cris}-linear map

$$D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \longrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}) \otimes_{K_0} B_{\mathrm{cris}} \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} B_{\mathrm{cris}}) \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$$

is Γ_K-equivariant and bijective. Moreover, this map is compatible with the natural Frobenius endomorphisms on $D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}}$ and $V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$ induced by φ. Let us now consider the induced K-linear bijective map

$$D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K) \longrightarrow V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K).$$

It is straightforward to verify that this map is filtered. Hence by Proposition 2.3.10, it suffices to prove the bijectivity of the graded map

$$\mathrm{gr}\left(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)\right) \longrightarrow \mathrm{gr}\left(V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K)\right). \tag{3.5}$$

Proposition 2.4.3 and Proposition 3.2.9 show that V is Hodge-Tate with a natural isomorphism

$$\mathrm{gr}(D_{\mathrm{cris}}(V)_K) \cong \mathrm{gr}(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{HT}}(V)$$

We apply Proposition 2.3.9 and Proposition 3.1.10 to obtain canonical isomorphisms

$$\mathrm{gr}\left(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)\right) \cong \mathrm{gr}(D_{\mathrm{cris}}(V)_K) \otimes_K \mathrm{gr}(B_{\mathrm{cris}} \otimes_{K_0} K) \cong D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}},$$
$$\mathrm{gr}\left(V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K)\right) \cong V \otimes_{\mathbb{Q}_p} \mathrm{gr}(B_{\mathrm{cris}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}.$$

Now we identify the map (3.5) with the natural map

$$D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}} \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}$$

given by Theorem 1.2.1 and in turn establish the desired assertion as V is Hodge-Tate. □

Remark. In our proof of Proposition 3.2.12, the finiteness of $\dim_{\mathbb{Q}_p}(V)$ and $\dim_K(D_{\mathrm{cris}}(V))$ are crucial for applying Proposition 2.3.9.

Proposition 3.2.13. The functor D_{cris} with values in MF_K^{φ} is faithful and exact on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$.

Proof. Since the forgetful functor $\mathrm{MF}_K^{\varphi} \to \mathrm{Vect}_{K_0}$ is faithful, Proposition 1.2.2 implies that D_{cris} is faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$. Hence it remains to verify that D_{cris} is exact on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$. Consider an exact sequence of crystalline Γ_K-representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$

By Proposition 1.2.2, this sequence gives rise to an exact sequence of isocrystals

$$0 \longrightarrow D_{\mathrm{cris}}(U) \longrightarrow D_{\mathrm{cris}}(V) \longrightarrow D_{\mathrm{cris}}(W) \longrightarrow 0. \tag{3.6}$$

Moreover, we apply Proposition 3.2.9 to identify the induced sequence of filtered vector spaces

$$0 \longrightarrow D_{\mathrm{cris}}(U)_K \longrightarrow D_{\mathrm{cris}}(V)_K \longrightarrow D_{\mathrm{cris}}(W)_K \longrightarrow 0$$

with the exact sequence of filtered vector spaces

$$0 \longrightarrow D_{\mathrm{dR}}(U) \longrightarrow D_{\mathrm{dR}}(V) \longrightarrow D_{\mathrm{dR}}(W) \longrightarrow 0$$

given by Proposition 2.4.9. Therefore we deduce that the sequence (3.6) is exact in MF_K^{φ}. □

PROPOSITION 3.2.14. Given a crystalline Γ_K-representation V, every subquotient W of V is crystalline with $D_{\text{cris}}(W)$ naturally identified as a subquotient of $D_{\text{cris}}(V)$ in MF_K^{φ}.

PROOF. The assertion is evident by Proposition 1.2.3 and Proposition 3.2.13. □

PROPOSITION 3.2.15. Given two crystalline Γ_K-representations V and W, their tensor product $V \otimes_{\mathbb{Q}_p} W$ is crystalline with a natural K-filtered isomorphism of isocrystals

$$D_{\text{cris}}(V) \otimes_{K_0} D_{\text{cris}}(W) \cong D_{\text{cris}}(V \otimes_{\mathbb{Q}_p} W). \tag{3.7}$$

PROOF. Proposition 1.2.4 shows that $V \otimes_{\mathbb{Q}_p} W$ is crystalline and yields the desired isomorphism (3.7) as a K_0-linear bijection. Since the construction of the map (3.7) rests on the multiplication of B_{cris}, it is straightforward to verify that the map (3.7) is a K-filtered morphism of isocrystals. Moreover, we apply Proposition 3.2.9 to identify the induced map

$$D_{\text{cris}}(V)_K \otimes_K D_{\text{cris}}(W)_K \longrightarrow D_{\text{cris}}(V \otimes_{\mathbb{Q}_p} W)_K.$$

with the natural K-linear filtered isomorphism

$$D_{\text{dR}}(V) \otimes_K D_{\text{dR}}(W)_K \cong D_{\text{dR}}(V \otimes_{\mathbb{Q}_p} W)$$

given by Proposition 2.4.11. Now we deduce that the map (3.7) is a K-filtered isomorphism of isocrystals, thereby completing the proof. □

PROPOSITION 3.2.16. Given a crystalline Γ_K-representation V and a positive integer n, both $\wedge^n(V)$ and $\mathrm{Sym}^n(V)$ are crystalline with natural K-filtered isomorphisms of isocrystals

$$\wedge^n(D_{\text{cris}}(V)) \cong D_{\text{cris}}(\wedge^n(V)) \quad \text{and} \quad \mathrm{Sym}^n(D_{\text{cris}}(V)) \cong D_{\text{cris}}(\mathrm{Sym}^n(V)).$$

PROOF. Proposition 1.2.5 shows that $\wedge^n(V)$ and $\mathrm{Sym}^n(V)$ are crystalline. Moreover, Proposition 1.2.5 yields the desired isomorphisms as K_0-linear bijections. Proposition 3.2.14 and Proposition 3.2.15 imply that these maps are K-filtered isomorphisms of isocrystals. □

Example 3.2.17. Given a crystalline Γ_K-representation V, we have

$$\mu(D_{\text{cris}}(V(n))) = \mu(D_{\text{cris}}(V)) - n \quad \text{for each } n \in \mathbb{Z}$$

by Example 3.2.10, Proposition 3.2.15, and Proposition 3.2.16.

PROPOSITION 3.2.18. For every crystalline Γ_K-representation V, the dual representation $V^\vee$ is crystalline with a natural K-filtered perfect pairing of isocrystals

$$D_{\text{cris}}(V) \otimes_{K_0} D_{\text{cris}}(V^\vee) \cong D_{\text{cris}}(V \otimes_{\mathbb{Q}_p} V^\vee) \longrightarrow D_{\text{cris}}(\mathbb{Q}_p).$$

PROOF. Proposition 1.2.7 shows that $V^\vee$ is crystalline and yields the desired pairing as a K_0-linear perfect pairing. This pairing is a K-filtered morphism of isocrystals over K_0 by Proposition 3.2.15 and thus gives rise to a K-filtered bijective morphism of isocrystals

$$D_{\text{cris}}(V)^\vee \longrightarrow D_{\text{cris}}(V^\vee). \tag{3.8}$$

Moreover, we apply Proposition 3.2.9 to identify the induced K-linear filtered map

$$D_{\text{cris}}(V)_K^\vee \longrightarrow D_{\text{cris}}(V^\vee)_K$$

with the natural K-linear filtered isomorphism

$$D_{\text{dR}}(V) \cong D_{\text{dR}}(V^\vee)$$

given by Proposition 2.4.14. Now we deduce that the map (3.8) is a K-filtered isomorphism of isocrystals, thereby completing the proof. □

Remark. Proposition 2.3.8 and Proposition 3.2.18 together show that the canonical isomorphism $V \cong (V^\vee)^\vee$ induces a natural K-filtered isomorphism $D_{\text{cris}}(V) \cong (D_{\text{cris}}(V)^\vee)^\vee$.

For the rest of this chapter, we generally write φ for a map naturally induced by the Frobenius endomorphism on B_{cris}. In order to discuss some additional properties of crystalline representations and the functor D_{cris}, we state the following remarkable result without a proof.

Theorem 3.2.19 (Fontaine [**Fon94a**]). The ring $B_e = B_{\mathrm{cris}}^{\varphi=1}$ fits into a natural exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow B_{\mathrm{dR}}/B_{\mathrm{dR}}^{+} \longrightarrow 0.$$

Remark. We will present a proof of Theorem 3.2.19 in Chapter IV.

Lemma 3.2.20. The ring $B_e = B_{\mathrm{cris}}^{\varphi=1}$ yields an identification

$$B_e \cap \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} K) = B_e \cap B_{\mathrm{dR}}^{+} = \mathbb{Q}_p.$$

Proof. By Proposition 3.1.9 and Theorem 3.2.19, we find

$$B_e \cap \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} K) \subseteq B_e \cap \mathrm{Fil}^0(B_{\mathrm{dR}}) = B_e \cap B_{\mathrm{dR}}^{+} = \mathbb{Q}_p.$$

Hence we obtain the desired identification as both B_e and $\mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} K)$ contain $\mathbb{Q}_p$. □

Proposition 3.2.21. Every crystalline Γ_K-representation V admits canonical isomorphisms

$$\begin{aligned} V &\cong (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \cap \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)\right) \\ &\cong (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \cap \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K B_{\mathrm{dR}}\right). \end{aligned}$$

Proof. Proposition 3.2.12 yields a natural Γ_K-equivariant isomorphism

$$D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$$

which is compatible with the Frobenius endomorphisms. Moreover, this isomorphism induces a canonical filtered isomorphism

$$D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{cris}} \otimes_{K_0} K),$$

which in turn gives rise to a natural filtered isomorphism

$$D_{\mathrm{cris}}(V)_K \otimes_K B_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}.$$

Therefore we obtain canonical isomorphisms

$$\begin{aligned} (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} &\cong V \otimes_{\mathbb{Q}_p} B_e, \\ \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K (B_{\mathrm{cris}} \otimes_{K_0} K)\right) &\cong V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^0(B_{\mathrm{cris}} \otimes_{K_0} K), \\ \mathrm{Fil}^0\left(D_{\mathrm{cris}}(V)_K \otimes_K B_{\mathrm{dR}}\right) &\cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^{+}. \end{aligned}$$

Now the desired assertion follows from Lemma 3.2.20. □

Theorem 3.2.22 (Fontaine [**Fon94b**]). The functor D_{cris} with values in MF_K^{φ} is exact and fully faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$.

Proof. By Proposition 3.2.13, we only need to prove that D_{cris} is full on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$. Let V and W be arbitrary crystalline Γ_K-representations. Consider an arbitrary morphism $f : D_{\mathrm{cris}}(V) \to D_{\mathrm{cris}}(W)$ in MF_K^{φ}. Proposition 3.2.12 yields a Γ_K-equivariant B_{cris}-linear map

$$V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}} \cong D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}} \xrightarrow{f \otimes 1} D_{\mathrm{cris}}(W) \otimes_{K_0} B_{\mathrm{cris}} \cong W \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}.$$

Moreover, Proposition 3.2.21 implies that this map restricts to a $\mathbb{Q}_p$-linear map $\phi : V \to W$. Now we identify f with the restriction of $\phi \otimes 1$ on $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K}$ under the identification

$$(V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K} \cong (D_{\mathrm{cris}}(V) \otimes_{K_0} B_{\mathrm{cris}})^{\Gamma_K} \cong D_{\mathrm{cris}}(V)$$

and in turn deduce that f corresponds to ϕ under the functor D_{cris}. □

PROPOSITION 3.2.23. Let V be a p-adic Γ_K-representation and L be a finite unramified extension of K with residue field l. Denote by L_0 the fraction field of $W(l)$.

(1) There exists an L-filtered isomorphism of isocrystals
$$D_{\mathrm{cris},K}(V) \otimes_{K_0} L_0 \cong D_{\mathrm{cris},L}(V)$$
where we set $D_{\mathrm{cris},K}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K}$ and $D_{\mathrm{cris},L}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_L}$.

(2) V is crystalline if and only if it is crystalline as a Γ_L-representation.

PROOF. Lemma 2.4.16 shows that L is a p-adic field. Moreover, L and L_0 are respectively Galois over K and K_0 with natural isomorphisms
$$\mathrm{Gal}(L/K) \cong \mathrm{Gal}(L_0/K_0) \cong \mathrm{Gal}(l/k).$$
Hence we find
$$D_{\mathrm{cris},K}(V) = D_{\mathrm{cris},L}(V)^{\mathrm{Gal}(L/K)} = D_{\mathrm{cris},L}(V)^{\mathrm{Gal}(L_0/K_0)}$$
and in turn apply Lemma 2.4.16 to obtain a natural bijective L_0-linear map
$$D_{\mathrm{cris},K}(V) \otimes_{K_0} L_0 \longrightarrow D_{\mathrm{cris},L}(V). \tag{3.9}$$
This map is evidently a morphism of isocrystals. In addition, by Proposition 2.4.17 and Proposition 3.2.9, the map (3.9) induces an L-linear filtered isomorphism
$$(D_{\mathrm{cris},K}(V) \otimes_{K_0} K) \otimes_K L \cong D_{\mathrm{cris},L}(V) \otimes_{L_0} L.$$
We deduce that the map (3.9) is an L-filtered isomorphism of isocrystals and consequently establish statement (1). Statement (2) is an immediate consequence of statement (1). ☐

Remark. We can show that Proposition 3.2.23 remains valid for $L = \widehat{K^{\mathrm{un}}}$ by the remark following Lemma 2.4.16. Hence every p-adic Γ_K-representation with a trivial I_K-action is crystalline.

Example 3.2.24. Given a continuous character $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ such that $\eta(I_K)$ is nontrivially finite, we assert that $\mathbb{Q}_p(\eta)$ is not crystalline. Since we have
$$D_{\mathrm{cris}}(\mathbb{Q}_p(\eta)) = (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K} \subseteq (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{I_K},$$
it suffices to show that $(\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{I_K}$ vanishes. Let us take a finite Galois extension L of K with $\eta(I_L)$ being trivial. By Proposition 3.2.23, we may replace K with an unramified extension to assume that L is totally ramified over K. Theorem 3.1.14 shows that the fraction field $\widehat{K_0^{\mathrm{un}}}$ of $W(\overline{k})$ admits a natural isomorphism $B_{\mathrm{cris}}^{I_L} \cong \widehat{K_0^{\mathrm{un}}} \cong B_{\mathrm{cris}}^{I_K}$, which in particular implies that $\mathrm{Gal}(L/K) \cong I_K/I_L$ acts trivially on $\widehat{K_0^{\mathrm{un}}}$. Meanwhile, we obtain the identities $\mathbb{Q}_p(\eta)^{I_L} = \mathbb{Q}_p(\eta)$ and $\mathbb{Q}_p(\eta)^{\mathrm{Gal}(L/K)} = 0$ respectively from the triviality of $\eta(I_L)$ and the nontriviality of $\eta(I_K)$. Hence we find
$$\begin{aligned}(\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{I_K} &= \left((\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{I_L}\right)^{\mathrm{Gal}(L/K)} = \left(\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}^{I_L}\right)^{\mathrm{Gal}(L/K)} \\ &\cong \left(\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} \widehat{K_0^{\mathrm{un}}}\right)^{\mathrm{Gal}(L/K)} = \mathbb{Q}_p(\eta)^{\mathrm{Gal}(L/K)} \otimes_{\mathbb{Q}_p} \widehat{K_0^{\mathrm{un}}} = 0\end{aligned}$$
as desired.

Remark. Example 3.2.24 shows that Proposition 3.2.23 fails for a ramified extension L of K. Fontaine interpreted this fact as an indication for a tidy connection between crystalline representations and abelian varieties with good reduction. In fact, an abelian variety A over K has good reduction if and only if the rational Tate module $V_p(A[p^\infty])$ is crystalline, as proved by Coleman-Iovita [**CI99**] and Breuil [**Bre00**]. Hence, in light of Theorem 1.1.2 in Chapter I, we may regard crystalline representations as p-adic counterparts of unramified representations.

For the rest of this section, we assume for simplicity that K is a finite extension of $\mathbb{Q}_p$.

LEMMA 3.2.25. Given a unit $u \in W(\overline{k})$ and an integer $r > 0$, there exists a unit $v \in W(\overline{k})$ with $\varphi^r(v) = uv$.

PROOF. For each $\alpha \in W(\overline{k})$, we denote by $\overline{\alpha}$ its image in $\overline{k} \cong W(\overline{k})/pW(\overline{k})$. Since φ^r is an isometry by construction, it suffices to present a sequence $(v_n) \in W(\overline{k})^\times$ with

$$\varphi^r(v_n) \in uv_n + p^{n+1}W(\overline{k}) \quad \text{and} \quad v_{n+1} - v_n \in p^{n+1}W(\overline{k}).$$

We take $v_0 \in W(\overline{k})^\times$ with $\overline{v_0}^{p^r} = \overline{uv_0}$ and inductively construct v_n for each $n \geq 1$. In fact, as we have $\varphi^r(v_{n-1}) = uv_{n-1} + p^n\alpha_n$ for some $\alpha_n \in W(\overline{k})$, we choose $\beta_n \in W(\overline{k})$ with $\overline{\beta_n}^{p^r} = \overline{u}\overline{\beta_n} - \overline{\alpha_n}$ and set $v_n := v_{n-1} + p^n\beta_n$ to find $\varphi^r(v_n) \in uv_n + p^{n+1}W(\overline{k})$ as desired. □

PROPOSITION 3.2.26. Let $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ be a continuous character with $\eta(I_K)$ being trivial.

(1) The p-adic Γ_K-representation $\mathbb{Q}_p(\eta)$ is crystalline with a Hodge-Tate weight 0.
(2) The isocrystal $D_{\text{cris}}(\mathbb{Q}_p(\eta))$ over K_0 has rank 1 and degree 0.

PROOF. Let us write r for the degree of k over $\mathbb{F}_p$ and take an element $\widetilde{\sigma} \in \Gamma_K$ whose image in $\Gamma_k \cong \Gamma_K/I_K$ is the p^r-th power map on $\overline{k}$. We note that η takes values in $\mathbb{Z}_p^\times$ by continuity and apply Lemma 3.2.25 to obtain an element $u \in W(\overline{k})^\times$ with $\varphi^r(u) = \eta(\widetilde{\sigma})^{-1}u$. The element $1 \otimes u \in \mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\text{cris}}$ is I_K-invariant by construction and is $\widetilde{\sigma}$-invariant as $\widetilde{\sigma}$ acts on $W(\overline{k})$ via φ^r. Since the group generated by $\widetilde{\sigma}$ has a dense image in $\Gamma_k \cong \Gamma_K/I_K$, we see that $D_{\text{cris}}(\mathbb{Q}_p(\eta)) = (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_K}$ contains $1 \otimes u$. Hence Theorem 1.2.1 shows that $\mathbb{Q}_p(\eta)$ is crystalline, which in particular implies that $D_{\text{cris}}(\mathbb{Q}_p(\eta))$ has rank 1. Moreover, $D_{\text{cris}}(\mathbb{Q}_p(\eta))$ has degree 0 as both u and $\varphi(u)$ are units in $W(\overline{k})$. Now we use Theorem 1.1.7, Proposition 2.4.3, and Proposition 3.2.9 to find that $\mathbb{Q}_p(\eta)$ has a Hodge-Tate weight 0, thereby completing the proof. □

Remark. While our proof of Proposition 3.2.26 relies on the assumption that K is a finite extension of $\mathbb{Q}_p$, Proposition 3.2.26 holds without the assumption as explained in the notes of Brinon-Conrad [**BC**, Lemma 8.3.3].

PROPOSITION 3.2.27. Let $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ be a continuous character.

(1) $\mathbb{Q}_p(\eta)$ is de Rham if and only if $(\eta\chi^n)(I_K)$ is finite for some $n \in \mathbb{Z}$.
(2) $\mathbb{Q}_p(\eta)$ is crystalline if and only if $(\eta\chi^n)(I_K)$ is trivial for some $n \in \mathbb{Z}$.

PROOF. Let us begin with statement (1). If $\mathbb{Q}_p(\eta)$ is de Rham, $(\eta\chi^n)(I_K)$ is finite for some $n \in \mathbb{Z}$ by Proposition 1.1.13 and Proposition 2.4.3. For the converse, we now assume that $(\eta\chi^n)(I_K)$ is finite for some $n \in \mathbb{Z}$. Take a finite extension L of K such that $(\eta\chi^n)(I_L)$ is trivial. Proposition 3.2.9 and Proposition 3.2.26 together show that $\mathbb{Q}_p(\eta\chi^n)$ is de Rham as a Γ_L-representation. Hence we deduce from Lemma 2.4.6 and Proposition 2.4.17 that $\mathbb{Q}_p(\eta)$ is de Rham as desired.

It remains to establish statement (2). If $(\eta\chi^n)(I_K)$ is trivial for some $n \in \mathbb{Z}$, we see by Proposition 3.2.26 that $\mathbb{Q}_p(\eta\chi^n)$ is crystalline and in turn deduce from Lemma 1.2.10 that $\mathbb{Q}_p(\eta)$ is also crystalline. For the converse, we henceforth assume that $\mathbb{Q}_p(\eta)$ is crystalline. Proposition 3.2.9 and statement (1) together imply that $(\eta\chi^n)(I_K)$ is finite for some $n \in \mathbb{Z}$. Meanwhile, Lemma 1.2.10 shows that $\mathbb{Q}_p(\eta\chi^n)$ is crystalline. Hence we find by Example 3.2.24 that $(\eta\chi^n)(I_K)$ is trivial, thereby completing the proof. □

Remark. By Proposition 1.1.13 and Proposition 3.2.27, being Hodge-Tate and being de Rham are equivalent for $\mathbb{Q}_p(\eta)$.

3.3. Admissible filtered isocrystals

In this subsection, we study filtered isocrystals over K which arise from crystalline Γ_K-representations.

Definition 3.3.1. Let D be a filtered isocrystal over K.

(1) If D is nonzero, we define its *filtration degree* to be the unique integer $\deg^\bullet(D)$ with $\mathrm{gr}^{\deg^\bullet(D)}(\det(D_K)) \neq 0$.

(2) We say that D is *weakly admissible* if every nonzero filtered subisocrystal D' of D satisfies the inequality $\deg^\bullet(D') \leq \deg(D')$ with equality for $D' = D$.

(3) We say that D is *admissible* if it admits an isomorphism $D \simeq D_{\mathrm{cris}}(V)$ for some crystalline Γ_K-reprsentation V.

Remark. Theorem 1.2.22 implies that the category of Γ_K-representation is equivalent to the category of admissible filtered isocrystals over K.

Lemma 3.3.2. The ring $B_e = B_{\mathrm{cris}}^{\varphi=1}$ has a trivial intersection with $\mathrm{Fil}^n(B_{\mathrm{dR}})$ for every $n > 0$.

Proof. We have $\mathbb{Q}_p \cap \mathrm{Fil}^n(B_{\mathrm{dR}}) = 0$ as all nonzero elements of $\mathbb{Q}_p$ are units in B_{dR}^+. Hence the desired assertion follows from Lemma 3.2.20. □

Proposition 3.3.3. Every admissible filtered isocrystal D over K is weakly admissible.

Proof. Since the assertion is evident for $D = 0$, we may assume that D is nonzero. Let us take a crystalline Γ_K-representation V with a K-filtered isomorphism of isocrystals

$$D \simeq D_{\mathrm{cris}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}})^{\Gamma_K}. \tag{3.10}$$

Proposition 3.2.16 shows that $\det(V)$ is crystalline with $D_{\mathrm{cris}}(\det(V)) \simeq \det(D)$. We see that the I_K-action is trivial on $\det(V)(n)$ for some $n \in \mathbb{Z}$ by Proposition 3.2.27 and in turn find

$$\deg(D_{\mathrm{cris}}(\det(V)(n))) = \deg^\bullet(D_{\mathrm{cris}}(\det(V(n)))) = 0$$

by Proposition 3.2.26. Since we have $\deg(D) = \deg(\det(D))$ and $\deg^\bullet(D) = \deg^\bullet(\det(D))$, we use Example 2.4.12 and Example 3.2.17 to obtain the equality

$$\deg(D) = \deg(D_{\mathrm{cris}}(\det(V))) = \deg^\bullet(D_{\mathrm{cris}}(\det(V))) = \deg^\bullet(D).$$

Let us now consider an arbitrary nonzero filtered subisocrystal D' of D. For notational simplicity, we write $d := \deg(D')$ and $d^\bullet := \deg^\bullet(D')$. We wish to prove the inequality $d^\bullet \leq d$. Proposition 3.2.16 shows that $\wedge^{\mathrm{rk}(D')} D$ is admissible. Since $\det(D') = \wedge^{\mathrm{rk}(D')} D'$ is naturally a filtered subisocrystal of $\wedge^{\mathrm{rk}(D')} D$ with $\deg(\det(D')) = d$ and $\deg^\bullet(\det(D')) = d^\bullet$, we may replace D' and D respectively by $\det(D')$ and $\wedge^{\mathrm{rk}(D')} D$ to assume that D' has rank 1. Let us take a K_0-basis element e of D' and a $\mathbb{Q}_p$-basis (v_i) of V. The isomorphism (3.10) yields a relation $e = \sum v_i \otimes b_i$ with $b_i \in B_{\mathrm{cris}}$. We take $u \in W(k)^\times$ with $\varphi(e) = p^d ue$ and use the identity $\varphi(e) = \sum v_i \otimes \varphi(b_i)$ to find $\varphi(b_i) = p^d u b_i$ for each b_i. Meanwhile, as we have $\mathrm{Fil}^{d^\bullet}(D'_K) \neq 0$ and $\mathrm{Fil}^{d^\bullet+1}(D'_K) = 0$, we see that $e = \sum v_i \otimes b_i$ lies in $V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^{d^\bullet}(B_{\mathrm{dR}})$ and in turn deduce that each b_i lies in $\mathrm{Fil}^{d^\bullet}(B_{\mathrm{dR}})$. Now we choose $v \in W(\overline{k})^\times$ with $\varphi(v) = u^{-1} v$ by Lemma 3.2.25 and observe that each $b_i t^{-d} v$ belongs to $B_e \cap \mathrm{Fil}^{d^\bullet - d}(B_{\mathrm{dR}})$. The desired inequality $d^\bullet \leq d$ follows from Lemma 3.3.2 and the fact that e is nonzero. □

Remark. As we will see in Chapter IV, the converse of Proposition 3.3.3 holds; in other words, a filtered isocrystal over K is admissible if and only if it is weakly admissible. Hence the category of crystalline p-adic Γ_K-representations is equivalent to the category of weakly admissible filtered isocrystals over K.

Definition 3.3.4. Let D be a filtered isocrystal over K.

(1) Given an integer n, the *n-fold Tate twist* of D is $D(n) := D \otimes_{K_0} D_{\mathrm{cris}}(\mathbb{Q}_p(n))$.
(2) A *Hodge-Tate weight* of D is an integer m with $\mathrm{gr}^m(D_K) \neq 0$.

Example 3.3.5. Let V be a crystalline Γ_K-representation. Proposition 3.2.15 yields a natural K-filtered isomorphism of isocrystals

$$D_{\mathrm{cris}}(V)(n) \cong D_{\mathrm{cris}}(V(n)) \quad \text{for each } n \in \mathbb{Z}.$$

Moreover, Proposition 2.4.4 and Proposition 3.2.9 together show that the Hodge-Tate weights of $D_{\mathrm{cris}}(V)$ coincides with the Hodge-Tate weights of V. For each Hodge-Tate weight m of V, its multiplicity is equal to $\mathrm{gr}^m(D_{\mathrm{cris}}(V)_K)$.

Remark. If V is not crystalline, we can still show that there exists a natural K-filtered isomorphism of isocrystals $D_{\mathrm{cris}}(V)(n) \cong D_{\mathrm{cris}}(V(n))$ for each $n \in \mathbb{Z}$. On the other hand, if V is not crystalline, the Hodge-Tate weights of $D_{\mathrm{cris}}(V)$ are in general not equal to the Hodge-Tate weights of V; indeed, it is possible that $D_{\mathrm{cris}}(V)$ vanishes and has no Hodge-Tate weights at all.

PROPOSITION 3.3.6. Let D be a nonzero filtered isocrystal over K and n be an integer.

(1) $D(n)$ is a filtered isocrystal over K with the Frobenius automorphism $p^{-n}\varphi_D$ and
$$\mathrm{Fil}^m(D(n)_K) = \mathrm{Fil}^{m+n}(D_K) \quad \text{for each } m \in \mathbb{Z}.$$
(2) $D(n)$ satisfies the equalities
$$\deg(D(n)) = \deg(D) - n\,\mathrm{rk}(D) \quad \text{and} \quad \deg^\bullet(D(n)) = \deg^\bullet(D) - n\,\mathrm{rk}(D).$$

PROOF. Statement (1) is straightforward to verify using Example 3.2.10. Statement (2) is an immediate consequence of statement (1). □

LEMMA 3.3.7. Let D be a filtered isocrystal over K and n be an integer.

(1) D is weakly admissible if and only if its Tate twist $D(n)$ is weakly admissible.
(2) D is admissible if and only if its Tate twist $D(n)$ is admissible.

PROOF. Every filtered subisocrystal D' of D yields a filtered subisocrystal $D'(n)$ of $D(n)$. Similarly, every filtered subisocrystal D'' of $D(n)$ yields a filtered subisocrystal $D''(-n)$ of D. Hence we deduce statement (1) from Proposition 3.3.6. In addition, we obtain statement (2) by Lemma 1.2.10 and Example 3.3.5. □

PROPOSITION 3.3.8. If a filtered isocrystal D over K has a Hodge-Tate weight n, its Tate twist $D(m)$ has a Hodge-Tate weight $n - m$.

PROOF. The assertion is evident by Proposition 3.3.6. □

PROPOSITION 3.3.9. A filtered isocrystal D over K with Hodge-Tate weights $m_1, \cdots, m_r$ satisfies the equality

$$\deg^\bullet(D) = \sum_{i=1}^{r} m_i \dim_K \mathrm{gr}^{m_i}(D_K).$$

PROOF. Since D_K is finite dimensional over K, we can construct a K-basis $(e_{i,j})$ of D_K such that each $\mathrm{Fil}^n(D_K)$ has a K-basis $(e_{i,j})_{i \geq n}$; indeed, we take $m \in \mathbb{Z}$ with $\mathrm{Fil}^m(D_K) = 0$ and inductively extend a K-basis for each $\mathrm{Fil}^n(D_K)$ to a K-basis for $\mathrm{Fil}^{n-1}(D_K)$. Hence the desired assertion is straightforward to verify by Proposition 2.3.6. □

PROPOSITION 3.3.10. Given a filtered isocrystal D over K, every nonzero filtered subisocrystal D' of D yields a filtered subisocrystal $\widetilde{D'}$ of D with the following properties:

(i) $\widetilde{D'}$ contains D' as a filtered subisocrystal and satisfies the relations

$$\mathrm{rk}(D') = \mathrm{rk}(\widetilde{D'}), \quad \deg(D') = \deg(\widetilde{D'}), \quad \deg^\bullet(D') \leq \deg^\bullet(\widetilde{D'}).$$

(ii) $\widetilde{D'}$ gives rise to a short exact sequence of filtered isocrystals

$$0 \longrightarrow \widetilde{D'} \longrightarrow D \longrightarrow D/\widetilde{D'} \longrightarrow 0.$$

PROOF. Take $\widetilde{D'}$ to be the isocrystal D' with the filtration on $\widetilde{D'}_K$ given by

$$\mathrm{Fil}^n(\widetilde{D'}_K) := \mathrm{Fil}^n(D_K) \cap D'_K \quad \text{ for each } n \in \mathbb{Z}.$$

We obtain property (i) by observing that the identity map on D' induces a K-filtered injective morphism of isocrystals $D' \hookrightarrow \widetilde{D'}$. In addition, we find

$$\mathrm{Fil}^n\left((D/\widetilde{D'})_K\right) = \mathrm{Fil}^n(D_K)/\big(\mathrm{Fil}^n(D_K) \cap \widetilde{D'}_K\big) = \mathrm{Fil}^n(D_K)/\,\mathrm{Fil}^n(\widetilde{D'}) \quad \text{ for each } n \in \mathbb{Z}$$

and in turn verify property (ii). □

Remark. In general, a quotient of D by D' does not necessarily exist in the category of filtered isocrystals since the category of filtered K-vector spaces is not abelian.

Definition 3.3.11. For a filtered isocrystal D over K with a nonzero filtered subisocrystal D', we refer to the filtered isocrystal $\widetilde{D'}$ given by Proposition 3.3.10 as the *saturation* of D' in D.

PROPOSITION 3.3.12. Given a short exact sequence of nonzero filtered isocrystals over K

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0,$$

we have the equalities

$$\deg(D) = \deg(D') + \deg(D'') \quad \text{ and } \quad \deg^\bullet(D) = \deg^\bullet(D') + \deg^\bullet(D'').$$

PROOF. The assertion is evident as we have a natural K-filtered isomorphism of isocrystals

$$\det(D) \cong \det(D') \otimes_{K_0} \det(D'')$$

by a standard fact stated in the Stacks project [**Sta**, Tag 0B38]. □

PROPOSITION 3.3.13. Let D be a nonzero filtered isocrystal over K.

(1) $D^\vee$ satisfies the equalities $\deg(D^\vee) = -\deg(D)$ and $\deg^\bullet(D^\vee) = -\deg^\bullet(D)$.

(2) D is weakly admissible if and only if $D^\vee$ is weakly admissible.

PROOF. For statement (1), the first equality is evident by Lemma 2.3.17 in Chapter II while the second equality follows from Proposition 2.3.9 and Proposition 3.3.9. Let us now consider statement (2). Since we have a K-filtered isomorphism of isocrystals $D \cong (D^\vee)^\vee$ by Propositoin 2.3.8, it suffices to prove that $D^\vee$ is weakly admissible when D is weakly admissible. Take an arbitrary filtered subisocrystal D' of $D^\vee$. Its saturation $\widetilde{D'}$ in $D^\vee$ gives rise to a short exact sequence of filtered isocrystals

$$0 \longrightarrow (D^\vee/\widetilde{D'})^\vee \longrightarrow D \longrightarrow \widetilde{D'}^\vee \longrightarrow 0.$$

Hence we use Proposition 3.3.12 and statement (1) to find

$$\begin{aligned}\deg^\bullet(D') \leq \deg^\bullet(\widetilde{D'}) &= -\deg^\bullet(\widetilde{D'}^\vee) = -\deg^\bullet(D) + \deg^\bullet\left((D^\vee/\widetilde{D'})^\vee\right)\\ &\leq -\deg(D) + \deg\left((D^\vee/\widetilde{D'})^\vee\right) = -\deg(\widetilde{D'}^\vee) = \deg(\widetilde{D'}) = \deg(D').\end{aligned}$$

We see that $D^\vee$ is weakly admissible as we have $\deg^\bullet(D^\vee) = \deg(D^\vee)$ by statement (1). □

PROPOSITION 3.3.14. Consider a short exact sequence of filtered isocrystals over K

$$0 \longrightarrow D' \longrightarrow D \longrightarrow D'' \longrightarrow 0.$$

(1) If D and D' are weakly admissible, D'' is also weakly admmissible.
(2) If D and D'' are weakly admissible, D' is also weakly admmissible.
(3) If D' and D'' are weakly admissible, D is also weakly admmissible.

PROOF. Statement (1) and statement (2) are equivalent by Proposition 3.3.13; indeed, we can deduce one from the other by replacing the given exact sequence with its dual sequence. For statement (2), let us assume that D and D'' are weakly admissible. Every nonzero filtered subisocrystal E' of D' satisfies the inequality $\deg^\bullet(E') \leq \deg(E')$ for being a filtered subisocrystal of D. Moreover, by Propsotion 3.3.12 we have

$$\deg^\bullet(D') = \deg^\bullet(D) - \deg^\bullet(D'') = \deg(D) - \deg(D'') = \deg(D').$$

Therefore D' is weakly admissible as asserted in statement (2).

We now assume for statement (3) that D' and D'' are weakly admissible. Let E be a nonzero filtered subisocrystal of D. Take the filtered subisocrystal $E' := D' \cap E$ of D' with

$$\mathrm{Fil}^n(E'_K) = \mathrm{Fil}^n(D'_K) \cap E_K \quad \text{for each } n \in \mathbb{Z}.$$

We note that E' gives rise to a filtered subisocrystal $E'' := E/E'$ of $D'' = D/D'$ and in turn apply Proposition 3.3.12 to find

$$\deg^\bullet(E) = \deg^\bullet(E') + \deg^\bullet(E'') \leq \deg(E') + \deg(E'') = \deg(E).$$

Moreover, for $E = D$ the inequality becomes an equality as we have $E' = D'$ and $E'' = D''$. Therefore D is weakly admissible as asserted in statement (3). □

Remark. Let WMF_K^φ denote the category of weakly admissible filtered isocrystals over K. Although $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ and WMF_K^φ are equivalent via an exact functor, their behaviors within the ambient categories $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ and MF_K^φ exhibit some differences as follows:

(1) WMF_K^φ is closed under taking extensions in MF_K^φ as noted in Proposition 3.3.14, whereas $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ is not closed under taking extensions in $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ by the remark after Example 1.1.12.
(2) WMF_K^φ turns out to be not closed under taking subquotients in MF_K^φ, whereas $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(\Gamma_K)$ is closed under taking subquotients in $\mathrm{Rep}_{\mathbb{Q}_p}(\Gamma_K)$ by Proposition 3.2.14.

PROPOSITION 3.3.15. Given filtered isocrystals D and D' over K, their direct sum $D \oplus D'$ is weakly admissible if and only if both D and D' are weakly admissible.

PROOF. If both D and D' are weakly admissible, their direct sum $D \oplus D'$ is weakly admissible by Proposition 3.3.14. For the converse, we now assume that $D \oplus D'$ is weakly admissible. Let E and E' respectively be nonzero filtered subisocrystals of D and D'. Since D and D' are filtered subisocrystals of $D \oplus D'$, we find

$$\deg^\bullet(E) \leq \deg(E) \quad \text{and} \quad \deg^\bullet(E') \leq \deg(E').$$

Moreover, for $E = D$ and $E' = D'$, these inequalities become equalities as we have

$$\deg^\bullet(D) + \deg^\bullet(D') = \deg^\bullet(D \oplus D') = \deg(D \oplus D') = \deg(D) + \deg(D')$$

by Proposition 3.3.12. We deduce that both D and D' are weakly admissible as desired, thereby completing the proof. □

Proposition 3.3.16. A filtered isocrystal D over K of rank 1 is admissible if and only if it is weakly admissible.

Proof. If D is admissible, it is weakly admissible by Proposition 3.3.3. For the converse, we henceforth assume that D is weakly admissible. By Proposition 3.3.6 and Lemma 3.3.7, we may replace D with $D(\deg(D))$ to also assume that D has degree 0. Let us take a K_0-basis element e of D and write $\varphi(e) = ue$ with $u \in W(k)^\times$. In addition, we denote by r the degree of k over $\mathbb{F}_p$ and choose an element $\widetilde{\sigma} \in \Gamma_K$ whose image in $\Gamma_k \cong \Gamma_K/I_K$ is the p^r-th power map on $\overline{k}$. We apply Lemma 3.2.25 to obtain an element $v \in W(\overline{k})^\times$ with $\varphi(v) = uv$ and find

$$\varphi^r(v) = \varphi^{r-1}(u) \cdots \varphi(u)uv.$$

We observe that $v \in W(\overline{k})^\times$ is φ^r-invariant and thus deduce see that $\varphi^r(v)v^{-1}$ lies in $\mathbb{Z}_p^\times$ for being φ-invariant. Since the group generated by $\widetilde{\sigma}$ has a dense image in $\Gamma_k \cong \Gamma_K/I_K$, there exists a continuous character $\eta : \Gamma_K \to \mathbb{Z}_p^\times$ with $\eta(I_K)$ being trivial and $\eta(\widetilde{\sigma}) = \varphi^r(v)^{-1}v$. The element $1 \otimes v \in \mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\text{cris}}$ is I_K-invariant by construction and is $\widetilde{\sigma}$-invariant as $\widetilde{\sigma}$ acts on $W(\overline{k})$ via φ^r. Hence Proposition 3.2.26 shows that $D_{\text{cris}}(\mathbb{Q}_p(\eta)) = (\mathbb{Q}_p(\eta) \otimes_{\mathbb{Q}_p} B_{\text{cris}})^{\Gamma_K}$ is admissible of rank 1 with a basis element $1 \otimes v$. Now we obtain a K-filtered isomorphism of isocrystals $D \simeq D_{\text{cris}}(\mathbb{Q}_p(\eta))$ which maps e to $1 \otimes v$ and in turn deduce that D is admissible as desired. □

Remark. While our proof of Proposition 3.3.16 relies on the assumption that K is a finite extension of $\mathbb{Q}_p$, Proposition 3.3.16 holds without the assumption as explained in the notes of Brinon-Conrad [**BC**, Lemma 8.3.3].

Proposition 3.3.17. A filtered isocrystal D over K of rank 1 with a Hodge-Tate weight m is weakly admissible if and only if it satisfies the following properties:

(i) Its Frobenius automorphism is the multiplication by $p^m u$ for some $u \in W(k)^\times$.

(ii) The filtration on D_K admits an identification

$$\text{Fil}^n(D_K) = \begin{cases} D_K & \text{for } n \leq m, \\ 0 & \text{for } n > m. \end{cases}$$

Proof. Since D has rank 1, we obtain the equality $\deg^\bullet(D) = m$ and consequently establish the desired assertion. □

Proposition 3.3.18. Let $\text{WMF}_K^{\varphi,\text{rk}=1}$ denote the set of isomorphism classes of weakly admissible filtered isocrystals over K of rank 1.

(1) $\text{WMF}_K^{\varphi,\text{rk}=1}$ is naturally an abelian group under tensor products.

(2) $\text{WMF}_K^{\varphi,\text{rk}=1}$ gives rise to a canonical surjective group homomorphism

$$\mathbb{Z} \times W(k)^\times \twoheadrightarrow \text{WMF}_K^{\varphi,\text{rk}=1}$$

whose kernel consists of the elements $(0, \sigma(u)u^{-1}))$ with $u \in W(k)^\times$.

Proof. Both statements are straightforward to verify by Proposition 3.3.17. □

Remark. Proposition 3.3.16 and Proposition 3.3.18 together provide an explicit classification of one-dimensional crystalline Γ_K-representations. This classification is particularly simple for $K = \mathbb{Q}_p$ as we have a natural isomorphism

$$\text{WMF}_{\mathbb{Q}_p}^{\varphi,\text{rk}=1} \cong \mathbb{Z} \times \mathbb{Z}_p^\times \cong \mathbb{Q}_p^\times$$

by the fact that the Frobenius automorphism on $\mathbb{Q}_p$ is the identity map.

Lemma 3.3.19. Let D be an isocrystal over $\mathbb{Q}_p$.

(1) The Frobenius automorphism φ_D is $\mathbb{Q}_p$-linear with $\nu(\det(\varphi_D)) = \deg(D)$.
(2) An element $v \in D$ spans a nonzero subisocrystal of D if and only if it is an eigenvector of φ_D.

Proof. Both statements are straightforward to verify. □

Definition 3.3.20. Let D be a filtered isocrystal over $\mathbb{Q}_p$ of rank 2.

(1) D is *normally weighted* if its Hodge-Tate weights are 0 and $\deg(D)$ with $0 \leq \deg(D)$.
(2) If D is normally weighted, its *Hodge subspace* refers to $\mathcal{H}(D) := \mathrm{Fil}^1(D)$.
(3) D is *indecomposable* if it does not admit a direct sum decomposition into filtered isocrystals over $\mathbb{Q}_p$ of rank 1.

Lemma 3.3.21. Every weakly admissible filtered isocrystal over $\mathbb{Q}_p$ of rank 2 admits a normally weighted Tate twist.

Proof. The assertion follows from Proposition 3.3.6 and Proposition 3.3.9. □

Proposition 3.3.22. A normally weighted filtered isocrystal D over $\mathbb{Q}_p$ of rank 2 is indecomposible if and only if φ_D does not admit an eigenbasis containing an element of $\mathcal{H}(D)$.

Proof. If φ_D admits linearly independent eigenvectors v_1 and v_2 with $v_2 \in \mathcal{H}(D)$, we obtain an isomorphism $D \simeq D_1 \oplus D_2$ where D_1 and D_2 are the filtered subisocrystals of D respectively spanned by v_1 and v_2 with $\deg^\bullet(D_1) = 0$ and $\deg^\bullet(D_2) = \deg(D)$. Conversely, if D admits an isomorphism $D \simeq D_1 \oplus D_2$ for some filtered subisocrystals D_1 and D_2 of rank 1 with $\deg(D_1) \leq \deg(D_2)$, we apply Lemma 3.3.19 to see that φ_D admits an eigenbasis given by nonzero elements $v_1 \in D_1$ and $v_2 \in D_2$ with $v_2 \in \mathcal{H}(D)$. □

Example 3.3.23. Let us consider the basis vectors $e_1 := (1, 0)$ and $e_2 := (0, 1)$ of $\mathbb{Q}_p^{\oplus 2}$.

(1) For $a, b \in \mathbb{Z}_p$ with $(a^2 - 4b)^{1/2} \notin \mathbb{Z}_p$, there exists a unique normally weighted filtered isocrystal $D^{\mathrm{irr}}_{a,b}$ over $\mathbb{Q}_p$ of rank 2 with

$$\varphi_{D^{\mathrm{irr}}_{a,b}} = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix} \qquad \text{and} \qquad \mathcal{H}(D^{\mathrm{irr}}_{a,b}) = \mathbb{Q}_p \nu(b) e_1,$$

which is indecomposable by Proposition 3.3.22.

(2) For nonzero $\lambda_1, \lambda_2 \in \mathbb{Z}_p$ with $\lambda_1\lambda_2 \in p\mathbb{Z}_p$, there exists a unique a normally weighted filtered isocrystal $D^{\mathrm{diag}}_{\lambda_1,\lambda_2}$ over $\mathbb{Q}_p$ of rank 2 with

$$\varphi_{D^{\mathrm{diag}}_{\lambda_1,\lambda_2}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \text{and} \qquad \mathcal{H}(D^{\mathrm{diag}}_{\lambda_1,\lambda_2}) = \mathbb{Q}_p(e_1 + e_2),$$

which is indecomposable by Proposition 3.3.22.

(3) For nonzero $\lambda \in \mathbb{Z}_p$, there exists a unique normally weighted filtered isocrystal D^{def}_λ over $\mathbb{Q}_p$ of rank 2 with

$$\varphi_{D^{\mathrm{def}}_\lambda} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \text{and} \qquad \mathcal{H}(D^{\mathrm{def}}_\lambda) = \mathbb{Q}_p \nu(\lambda) e_1,$$

which is indecomposable by Proposition 3.3.22.

Remark. $\mathcal{H}(D^{\mathrm{irr}}_{a,b})$ vanishes for $b \in \mathbb{Z}_p^\times$ and has a basis vector e_1 for $b \in p\mathbb{Z}_p$. Similarly, $\mathcal{H}(D^{\mathrm{def}}_\lambda)$ vanishes for $\lambda \in \mathbb{Z}_p^\times$ and has a basis vector e_1 for $\lambda \in p\mathbb{Z}_p$.

PROPOSITION 3.3.24. Let D be an indecomposable filtered isocrystal over $\mathbb{Q}_p$ of rank 2 with two distinct Hodge-Tate weights.

(1) When φ_D has no $\mathbb{Q}_p$-eigenvalues, D is weakly admissible if and only if it is isomorphic to a Tate twist of $D^{\text{irr}}_{a,b}$ for some $a \in \mathbb{Z}_p$ and $b \in p\mathbb{Z}_p$ with $(a^2-4b)^{1/2} \notin \mathbb{Z}_p$.
(2) When φ_D has two distinct $\mathbb{Q}_p$-eigenvalues, D is weakly admissible if and only if it is isomorphic to a Tate twist of $D^{\text{diag}}_{\lambda_1,\lambda_2}$ for some nonzero $\lambda_1, \lambda_2 \in \mathbb{Z}_p$ with $\lambda_1\lambda_2 \in p\mathbb{Z}_p$.
(3) When φ_D has a unique $\mathbb{Q}_p$-eigenvalue, D is weakly admissible if and only if it is isomorphic to a Tate twist of D^{def}_{λ} for some nonzero $\lambda \in p\mathbb{Z}_p$.

PROOF. By Lemma 3.3.7 and Lemma 3.3.21, we may assume without loss of generality that D is normally weighted. Since we have $\deg^\bullet(D) = \deg(D)$ by Proposition 3.3.9, the filtered isocrystal D is weakly admissible if and only if every nonzero filtered subisocrystal D' satisfies the inequality $\deg^\bullet(D') \leq \deg(D')$. Meanwhile, the eigenvalues of φ_D are nonzero as φ_D is an automorphism.

Let us first consider the case where φ_D admits no $\mathbb{Q}_p$-eigenvalues. Lemma 3.3.19 shows that D is irreducible and thus is weakly admissible. Take a nonzero element $e_1 \in \mathcal{H}(D)$ and set $e_2 := \varphi_D(e_1)$. We see that e_1 and e_2 form a $\mathbb{Q}_p$-basis for D, under which we may write

$$\varphi_D = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix} \qquad \text{for some } a, b \in \mathbb{Q}_p.$$

The characteristic polynomial of φ_D is $f(z) = z^2 - az + b$. Since f is irreducible over $\mathbb{Q}_p$, the element $a^2 - 4b \in \mathbb{Q}_p$ is a nonsquare. Moreover, we find $\nu(b) = \deg(D) > 0$ by Lemma 3.3.19 and in turn obtain the inequality $\nu(a) \geq 0$ by observing that the roots of the irreducible polynomial f over $\mathbb{Q}_p$ have the same valuation. Therefore we establish statement (1).

We now consider the case where φ_D has distinct nonzero $\mathbb{Q}_p$-eigenvalues λ_1 and λ_2. Choose $\mathbb{Q}_p$-basis vectors e_1 and e_2 for D with $\varphi_D(e_1) = \lambda_1 e_1$ and $\varphi_D(e_2) = \lambda_2 e_2$. Since $\mathcal{H}(D)$ contains neither e_1 nor e_2 by Proposition 3.3.22, we may replace e_1 and e_2 with their $\mathbb{Q}_p$-multiples to assume that $\mathcal{H}(D)$ contains $e_1 + e_2$. We have

$$\varphi_D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

under our basis and find $\nu(\lambda_1\lambda_2) = \deg(D) > 0$ by Lemma 3.3.19. Let us write D_i' for the filtered isocrystal over $\mathbb{Q}_p$ spanned by e_i with

$$\text{Fil}^n(D_i') = \text{Fil}^n(D) \cap D_i' \qquad \text{for each } n \in \mathbb{Z}.$$

For each D_i', we find $\deg^\bullet(D_i') = 0$ and $\deg(D_i') = \nu(\lambda_i)$. Meanwhile, for every filtered subisocrystal of D with rank 1, its saturation in D is D_1' or D_2' by Lemma 3.3.19. Hence D is weakly admissible if and only if we have $\nu(\lambda_i) \geq 0$ for each λ_i. Statement (2) is now evident.

It remains to consider the case where φ_D has a unique nonzero $\mathbb{Q}_p$-eigenvalue λ. Since φ_D is not a scalar multiplication by Proposition 3.3.22, we may write

$$\varphi_D = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with respect to some $\mathbb{Q}_p$-basis vectors e_1 and e_2 for D. Let D' be an arbitrary filtered subisocrystal of D with $\text{rk}(D') = 1$. Since we have $e_1 \in D'$ and $\deg(D') = \nu(\lambda) = \deg(D)/2$ by Lemma 3.3.19, the saturation $\widetilde{D'}$ of D' in D satisfies the inequality $\deg^\bullet(\widetilde{D'}) \leq \deg(\widetilde{D'})$ if and only if e_1 lies in $\mathcal{H}(D)$. Hence D is weakly admissible if and only if e_1 lies in $\mathcal{H}(D)$. Now we establish statement (3) and in turn complete the proof. □

Proposition 3.3.25. Let D be an indecomposable filtered isocrystal over $\mathbb{Q}_p$ of rank 2 with a unique Hodge-Tate weight.

(1) When φ_D has no $\mathbb{Q}_p$-eigenvalues, D is weakly admissible if and only if it is isomorphic to a Tate twist of $D_{a,b}^{\mathrm{irr}}$ for some $a \in \mathbb{Z}_p$ and $b \in \mathbb{Z}_p^\times$ with $(a^2-4b)^{1/2} \notin \mathbb{Z}_p$.
(2) When φ_D has a $\mathbb{Q}_p$-eigenvalue, D is weakly admissible if and only if it is isomorphic to a Tate twist of D_λ^{def} for some nonzero $\lambda \in \mathbb{Z}_p^\times$.

Proof. By Lemma 3.3.7 and Lemma 3.3.21, we may assume without loss of generality that D is normally weighted. Since we have $\deg^\bullet(D) = \deg(D)$ by Proposition 3.3.9, the filtered isocrystal D is weakly admissible if and only if every nonzero filtered subisocrystal D' satisfies the inequality $\deg^\bullet(D') \leq \deg(D')$. Meanwhile, the eigenvalues of φ_D are nonzero as φ_D is an automorphism.

Let us first consider the case where φ_D admits no $\mathbb{Q}_p$-eigenvalues. Lemma 3.3.19 shows that D is irreducible and thus is weakly admissible. Choose a nonzero element $e_1 \in D$ and set $e_2 := \varphi_D(e_1)$. We see that e_1 and e_2 form a $\mathbb{Q}_p$-basis for D, under which we may write

$$\varphi_D = \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix} \qquad \text{for some } a, b \in \mathbb{Q}_p.$$

The characteristic polynomial of φ_D is $f(z) = z^2 - az + b$. Since f is irreducible over $\mathbb{Q}_p$, the element $a^2 - 4b \in \mathbb{Q}_p$ is a nonsquare. Moreover, we find $\nu(b) = \deg(D) = 0$ by Lemma 3.3.19 and in turn obtain the inequality $\nu(a) \geq 0$ by observing that the roots of the irreducible polynomial f over $\mathbb{Q}_p$ have the same valuation. Therefore we establish statement (1).

We now consider the case where φ_D has a nonzero $\mathbb{Q}_p$-eigenvalue. Since D has a unique Hodge-Tate weight 0, its Hodge subspace $\mathcal{H}(D)$ must vanish. Proposition 3.3.22 implies that φ_D does not admit an eigenbasis, which means that φ_D has a unique nonzero eigenvalue λ and is not a scalar multiplication. Hence we may write

$$\varphi_D = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with respect to some $\mathbb{Q}_p$-basis vectors e_1 and e_2 for D. Let D' be an arbitrary filtered subisocrystal of D with $\mathrm{rk}(D') = 1$. For its saturation $\widetilde{D'}$ in D, we find $\deg^\bullet(\widetilde{D'}) = 0$ as D has a unique Hodge-Tate weight 0. Moreover, Lemma 3.3.19 shows that D' contains e_1 and satisfies the equality

$$\deg(D') = \nu(\lambda) = \deg(D)/2 = 0.$$

We deduce that D is weakly admissible and in turn establish statement (2). □

Remark. We have a complete classification for weakly admissible filtered isocrystals over $\mathbb{Q}_p$ of rank 2 by Proposition 3.3.15, Proposition 3.3.17, Proposition 3.3.24 and Proposition 3.3.25. In fact, up to Tate twists, the weakly admissible filtered isocrystals over $\mathbb{Q}_p$ of rank 2 are precisely the ones listed in Example 3.3.23 and the direct sums of weakly admissible filtered isocrystals over $\mathbb{Q}_p$ of rank 1 classified by Proposition 3.3.17. Since the category of weakly admissible filtered isocrystals over $\mathbb{Q}_p$ is equivalent to the category of crystalline p-adic $\Gamma_{\mathbb{Q}_p}$-representations as noted after Proposition 3.3.3, we obtain a complete classification for two-dimensional crystalline p-adic $\Gamma_{\mathbb{Q}_p}$-representations by Proposition 3.2.21.

If K is totally ramified over $\mathbb{Q}_p$, we can establish a similar classification for weakly admissible filtered isocrystals over K of rank 2, or equivalently for two-dimensional crystalline p-adic Γ_K-representations. In the general case, however, such a classification is very difficult to obtain. The main issue is that the Frobenius automorphism of an isocrystal over K_0 is not K_0-linear unless K is totally ramified over $\mathbb{Q}_p$.

Exercises

1. Let B be a $(\mathbb{Q}_p, \Gamma_K)$-regular ring and V be a p-adic Γ_K-representation of dimension d.
   (1) Show that $V \otimes_{\mathbb{Q}_p} B$ naturally gives rise to an element $[V]_B \in H^1(\Gamma_K, \mathrm{GL}_d(B))$.
   (2) Show that V is B-admissible if and only if $[V]_B \in H^1(\Gamma_K, \mathrm{GL}_d(B))$ is the distinguished element.

2. Let V be a p-adic Γ_K-representation.
   (1) Show that V is $\mathbb{C}_K$-admissibile if and only if it is Hodge-Tate with a unique Hodge-Tate weight 0.
   (2) Show that V is $\overline{K}$-admissible if and only if the Γ_K-action on V factors through a finite quotient.

   Hint. Use Theorem 1.2.1 and Lemma 2.4.16 along with the fact that the Γ_K-action on $\overline{K}$ is discrete.

3. Let A be an abelian variety over K of dimension g with good reduction.
   (1) Find the multiplicity for each Hodge-Tate weight of the étale cohomology $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$.
   (2) Prove that the I_K-action on $H^n_{\text{ét}}(A_{\overline{K}}, \mathbb{Q}_p)$ does not factor through a finite quotient.

Remark. The second part shows that the Neron-Ogg-Shafarevich criterion (Theorem 1.1.2 in Chapter I) fails for $\ell = p$.

4. Let L be a complete nonarchimedean field with valuation ν_L.
   (1) Prove that $L^\flat := \varprojlim_{x \mapsto x^p} L$ is naturally a perfect field of characteristic p which is complete with respect to a valuation induced by ν_L.
   (2) If the p-th power map on $\mathcal{O}_L/p\mathcal{O}_L$ is surjective, prove that the residue fields of L and $L^\flat$ are naturally isomorphic.
   (3) If L is a p-adic field, prove that $L^\flat$ is naturally isomorphic to the residue field of L.

Remark. The last part shows that the value groups of L and $L^\flat$ are not necessarily equal if L is not perfectoid.

5. Let ζ_{p^∞} and p^{1/p^∞} respectively denote the sets of p-power roots of 1 and p in $\overline{\mathbb{Q}_p}$.
   (1) Show that the p-adic completions of $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p(p^{1/p^\infty})$ are perfectoid fields.
   (2) Show that the p-adic completions of $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p(p^{1/p^\infty})$ are not isomorphic.
   (3) Show that the p-adic completions of $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p(p^{1/p^\infty})$ have isomorphic tilts.

   Hint. For the p-adic completion of of $\mathbb{Q}_p(\zeta_{p^\infty})$, establish an isomorphism
   $$\mathbb{Z}_p[\zeta_{p^\infty}] \cong \mathbb{Z}_p[u^{1/p^\infty}]/(1 + u + \cdots + u^{p-1})$$
   where u^{1/p^∞} denotes the set of p-power roots of the variable u.

6. In this exercise, we study sections of the map $\theta_{\mathrm{dR}}^{+} : B_{\mathrm{dR}}^{+} \twoheadrightarrow \mathbb{C}_K$.
   (1) Show that θ_{dR}^{+} admits a section $s_{\mathrm{dR}}^{+} : \mathbb{C}_K \to B_{\mathrm{dR}}^{+}$.

   Hint. Take a maximal subfield C of B_{dR}^{+}. Show that $\mathbb{C}_K$ is algebraic over $\theta_{\mathrm{dR}}^{+}(C)$ and use Hensel's lemma to find $\mathbb{C}_K = \theta_{\mathrm{dR}}^{+}(C)$.

   (2) Show that every section of θ_{dR}^{+} is neither continuous nor Γ_K-equivariant.

7. Let V and W be filtered vector spaces over a field L.
   (1) Show that $\mathrm{Hom}_L(V, W)$ is a filtered L-vector space where each $\mathrm{Fil}^n(\mathrm{Hom}_L(V, W))$ consists of the L-linear maps $f : V \to W$ sending each $\mathrm{Fil}^m(V)$ into $\mathrm{Fil}^{m+n}(W)$.
   (2) If V and W are finite dimensional, establish a natural filtered isomorphism
   $$\mathrm{Hom}_L(V, W) \cong V^\vee \otimes_L W.$$

8. Let L be an arbitrary field.
   (1) Find an L-linear filtered map $f : V \to W$ with the following properties:
      (i) The induced map $\mathrm{gr}(f) : \mathrm{gr}(V) \to \mathrm{gr}(W)$ is bijective.
      (ii) f is not a filtered isomorphism.
   (2) Find a bijective L-linear filtered map which is not a filtered isomorphism.

9. Consider a short exact sequence of p-adic Γ_K-representations
   $$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$
   (1) If U and W are Hodge-Tate with no common Hodge-Tate weights, prove that V is Hodge-Tate.
   (2) If U and W are de Rham with the Hodge-Tate weights of U being greater than all Hodge-Tate weights of W, prove that V is de Rham.

10. Let V and W be p-adic Γ_K-representations.
   (1) When V and W are Hodge-Tate, show that $D_{\mathrm{HT}}(V)$ and $D_{\mathrm{HT}}(W)$ are isomorphic if and only if V and W have the same Hodge-Tate weights with the same multiplicities.
   (2) When V and W are de Rham, show that $D_{\mathrm{dR}}(V)$ and $D_{\mathrm{dR}}(W)$ are isomorphic if and only if V and W have the same Hodge-Tate weights with the same multiplicities.

11. Let V be a p-adic Γ_K-representation and L be a finite extension of K.
   (1) Show that there exists a natural L-linear graded isomorphism
   $$D_{\mathrm{HT},K}(V) \otimes_K L \cong D_{\mathrm{HT},L}(V)$$
   where we set $D_{\mathrm{HT},K}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{\Gamma_K}$ and $D_{\mathrm{HT},L}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}})^{\Gamma_L}$.
   (2) Show that V is Hodge-Tate if and only if it is Hodge-Tate as a p-adic Γ_L-representation.

12. Prove that A_{inf}, A_{cris}, B_{cris}^{+}, and B_{dR}^{+} are not $(\mathbb{Q}_p, \Gamma_K)$-regular.

13. Consider the element $b := \dfrac{[\varepsilon^{1/p^2}] - 1}{[\varepsilon^{1/p}] - 1} \in B_{\text{dR}}$.
    (1) Show that b lies in $\text{Fil}^0(B_{\text{cris}}) = B_{\text{cris}} \cap B_{\text{dR}}^+$.
    Hint. Observe that b^{-1} lies in A_{inf} and show that $b/\varphi(b^{-1})$ lies in B_{cris}.
    (2) Show that b does not lie in B_{cris}^+.

14. Establish a natural isomorphism $\widehat{K^{\text{un}}} \cong K \otimes_{K_0} W(\overline{k})[1/p]$.

15. Consider the element $b_n := \dfrac{\xi^{p^n-1}}{(p^n-1)!} \in B_{\text{cris}}^+$ for each $n \geq 1$.
    (1) Show that the sequence (b_n) does not converge to 0 in B_{cris}^+.
    (2) Show that the sequence (b_n) converges to 0 in B_{cris}.
    Hint. Observe that the sequence (ξb_n) converges to 0 in B_{cris}.

16. Prove that every p-adic Γ_K-representation V yields a natural K-filtered isomorphism
$$D_{\text{cris}}(V)(n) \cong D_{\text{cris}}(V(n)) \quad \text{for each } n \in \mathbb{Z}.$$

17. Let $f : D \to D'$ be a bijective K-filtered morphism of isocrystals.
    (1) Prove the relations $\deg(D) = \deg(D')$ and $\deg^{\bullet}(D) \leq \deg^{\bullet}(D')$.
    (2) Prove that f is a K-filtered isomorphism if and only if we have $\deg^{\bullet}(D) = \deg^{\bullet}(D')$.

18. Let WMF_K^{φ} denote the category of weakly admissible filtered isocrystals over K.
    (1) Show that WMF_K^{φ} is not closed under taking subquotients in MF_K^{φ}.
    (2) Show that WMF_K^{φ} is abelian.

19. Assume that K is a finite extension of $\mathbb{Q}_p$.
    (1) Show that there exists a nonsplit extension of $\mathbb{Q}_p(1)$ by $\mathbb{Q}_p$.
    Hint. Use the local Tate duality to obtain the identification $H^1(\Gamma_K, \mathbb{Q}_p(-1)) \cong K$.
    (2) Show that every nonsplit extension of $\mathbb{Q}_p(1)$ by $\mathbb{Q}_p$ is not crystalline.

20. Assume that K is a finite extension of $\mathbb{Q}_p$.
    (1) Given $a, b \in \mathbb{Z}_p$ and an integer $r > 0$, prove that there exists an element $\lambda \in W(\overline{k})$ with $\varphi^{2r}(\lambda) + a\varphi^r(\lambda) + b\lambda = 0$.
    Hint. Write the desired relation in the form $\varphi^r(\varphi^r(\lambda) - \alpha) = \beta(\varphi^r(\lambda) - \alpha)$.
    (2) Prove that a p-adic Γ_K-representation V of dimension 2 is crystalline with a unique Hodge-Tate weight 0 if and only if the I_K-action on V is trivial.
    Hint. Adapt the arguments in Proposition 3.2.26 and Proposition 3.3.16, possibly by applying Proposition 3.2.21 and Proposition 3.3.25.

CHAPTER IV

The Fargues-Fontaine curve

1. Construction and geometric structures

In this section, we construct the algebraic Fargues-Fontaine curve and establish its fundamental properties. Our discussion involves extensions of many notions from Chapter III. The primary references for this section are the survey article of Fargues-Fontaine [**FF12**] and the lecture notes of Lurie [**Lur**].

Throughout this chapter, we let F be an algebraically closed perfectoid field of characteristic p with valuation ν_F. In addition, we denote by $\mathfrak{m}_F$ the maximal ideal of $\mathcal{O}_F$.

1.1. Untilts of a perfectoid field

In this subsection, we introduce and study untilts of the perfectoid field F. These objects serve as our main tools for defining and investigating the key objects in this section.

Definition 1.1.1. An *untilt* of F is a perfectoid field C together with a topological isomorphism $\iota_C : F \simeq C^\flat$ called the *tilting isomorphism* of C.

Example 1.1.2. The *trivial untilt* of F is the field F with the natural isomorphism $F \cong F^\flat$ given by Proposition 2.1.14 in Chapter III.

Remark. For a p-adic field K, the perfectoid field $\mathbb{C}_K^\flat$ turns out to be algebraically closed as we will prove in §3.1. Hence we may regard $\mathbb{C}_K$ as a distinguished untilt of $F = \mathbb{C}_K^\flat$.

Proposition 1.1.3. Every untilt C of F admits a unique valuation ν_C with $\nu_F(c) = \nu_C(c^\sharp)$ for each $c \in F$.

Proof. Choose a valuation ν on C. By Proposition 2.1.7 in Chapter III, there exists a valuation $\nu^\flat$ on F with $\nu^\flat(c) = \nu(c^\sharp)$ for every $c \in F$. Since ν_F and $\nu^\flat$ are equivalent, we have a group isomorphism $\delta : \mathbb{R} \to \mathbb{R}$ with $\delta(\nu^\flat(c)) = \nu_F(c)$ for every nonzero $c \in F$. Now we obtain a desired valuation ν_C by setting $\nu_C(x) = \delta(\nu(x))$ for each nonzero $x \in C$.

It remains to verify that such a valuation is unique. Let ν_C' be another valuation on C with $\nu_F(c) = \nu_C'(c^\sharp)$ for each $c \in F$. Take an arbitrary element $x \in C$. By Proposition 2.1.11 in Chapter III, we obtain an element $c \in F$ with

$$\nu_C(x) = \nu_F(c) = \nu_C(c^\sharp).$$

Now we write $x = uc^\sharp$ for some $u \in \mathcal{O}_C^\times$ and find

$$\nu_C'(x) = \nu_C'(c^\sharp) = \nu_F(c) = \nu_C(c^\sharp) = \nu_C(x).$$

We deduce that ν_C and ν_C' coincide, thereby completing the proof. □

Definition 1.1.4. Given an untilt C of F, we refer to the valuation ν_C on C given by Proposition 1.1.3 as the *normalized valuation* on C.

Proposition 1.1.5. Let C be an untilt of F and $f(z)$ be an irreducible monic polynomial of degree d over C. For every $x \in C$, there exists an element $y \in C$ with

$$\nu_C(x-y) \geq \nu_C(f(x))/d \quad \text{and} \quad \nu_C(f(y)) \geq \nu_C(p) + \nu_C(f(x)).$$

Proof. We may replace $f(z)$ with $f(z+x)$ to assume that x is zero. The irreducibility of f implies that $f(0)$ is nonzero. We wish to find an element $y \in C$ with

$$\nu_C(y) \geq \nu_C(f(0))/d \quad \text{and} \quad \nu_C(f(y)) \geq \nu_C(p) + \nu_C(f(0)).$$

Since F is algebraically closed, the multiplication by d is surjective on the value group of F. Hence Proposition 2.1.11 in Chapter III implies that the multiplication by d is also surjective on the value group of C. In particular, there exists an element $a \in C$ with $d\nu_C(a) = \nu_C(f(0))$. Now we aim to find an element $y \in C$ with

$$\nu_C(y/a) \geq 0 \quad \text{and} \quad \nu_C\left(f(a(y/a))/a^d\right) \geq \nu_C(p).$$

Let us take the monic irreducible polynomial $g(z) := f(az)/a^d$ over C. It suffices to present an element $w \in \mathcal{O}_C$ with $g(w) \in p\mathcal{O}_C$.

We assert that $g(z)$ is a polynomial over $\mathcal{O}_C$. Choose a finite Galois extension C' of C which contains all roots of $g(z)$. The valuation ν_C on C extends to a $\mathrm{Gal}(C'/C)$-equivariant valuation $\nu_{C'}$ on C'. Moreover, the roots of $g(z)$ have the same valuation for being in the same $\mathrm{Gal}(C'/C)$-orbit. Since we have $\nu_C(g(0)) = 0$, we see that each root of $g(z)$ has valuation 0 and consequently deduce that all coefficients of $g(z)$ lie in $\mathcal{O}_C$ as desired.

For each $b \in \mathcal{O}_C$, let us denote its image in $\mathcal{O}_C/p\mathcal{O}_C$ by $\bar{b}$. In addition, we write

$$g(z) = z^d + b_1 z^{d-1} + \cdots + b_d \quad \text{with } b_i \in \mathcal{O}_C.$$

Lemma 2.1.9 in Chapter III shows that each $b_i \in \mathcal{O}_C$ yields an element $c_i \in \mathcal{O}_F$ with $\overline{b_i} = \overline{c_i^\sharp}$. Since F is algebraically closed, there exists an element $\alpha \in \mathcal{O}_F$ with

$$\alpha^d + c_1\alpha^{d-1} + \cdots + c_d = 0.$$

Now we apply Proposition 2.1.10 in Chapter III to find

$$\overline{g(\alpha^\sharp)} = \overline{\alpha^\sharp}^d + \overline{b_1}\,\overline{\alpha^\sharp}^{d-1} + \cdots + \overline{b_d} = \overline{\alpha^\sharp}^d + \overline{c_1^\sharp}\,\overline{\alpha^\sharp}^{d-1} + \cdots + \overline{c_d^\sharp} = \overline{(\alpha^d + c_1\alpha^{d-1} + \cdots + c_d)^\sharp} = 0$$

and in turn complete the proof by taking $w = \alpha^\sharp$. □

Proposition 1.1.6. Every untilt C of F is algebraically closed.

Proof. If C has characteristic p, the assertion is evident as C is isomorphic to F by Proposition 2.1.14 in Chapter III. Let us henceforth assume that C has characteristic 0. Take an arbitrary monic irreducible polynomial $f(z)$ of degree d over C. We wish to show that $f(z)$ has a root in C. We may replace $f(z)$ by $p^{md}f(z/p^m)$ for some sufficiently large $m \in \mathbb{Z}$ to assume that $f(z)$ is a polynomial over $\mathcal{O}_C$. Since we have $\nu_C(f(0)) \geq 0$, we set $x_0 := 0$ and apply Proposition 1.1.5 to inductively construct a sequence (x_n) in C with

$$\nu_C(x_{n-1} - x_n) \geq (n-1)\nu_C(p)/d \quad \text{and} \quad \nu_C(f(x_n)) \geq n\nu_C(p) \quad \text{for each } n \geq 1.$$

The sequence (x_n) is Cauchy by construction and thus converges to an element $x \in C$. Now we obtain the identity

$$f(x) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = 0,$$

thereby completing the proof. □

Remark. Proposition 1.1.6 is a special case of the tilting equivalence for perfectoid fields.

Definition 1.1.7. The *infinitesimal period ring* associated to F is $A_{\text{inf}} = A_{\text{inf}}(F) := W(\mathcal{O}_F)$.

LEMMA 1.1.8. The ring A_{inf} is an integral domain.

PROOF. The assertion follows from Lemma 2.3.9 in Chapter II. □

Definition 1.1.9. Let ξ be an element in A_{inf}.

(1) We say that ξ is *primitive* if it has the form $\xi = [\varpi] - up$ with $\varpi \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$.
(2) We say that ξ is *strongly primitive* if it is primitive and not divisible by p.

PROPOSITION 1.1.10. An element $\xi \in A_{\text{inf}}$ with a Teichmüller expansion $\xi = \sum [c_n]p^n$ is primitive if and only if we have $\nu_F(c_0) > 0$ and $\nu_F(c_1) = 0$.

PROOF. The assertion follows from Proposition 2.3.6 in Chapter II. □

PROPOSITION 1.1.11. Given a strongly primitive element ξ in A_{inf}, the ring $A_{\text{inf}}/\xi A_{\text{inf}}$ is p-torsion free and p-adically complete.

PROOF. Lemma 1.1.8 implies that $A_{\text{inf}}/\xi A_{\text{inf}}$ is p-torsion free as ξ has a nonzero image in $A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$. Hence we only need to prove that $A_{\text{inf}}/\xi A_{\text{inf}}$ is p-adically complete. We write $\widehat{A_{\text{inf}}/\xi A_{\text{inf}}}$ for the p-adic completion of $A_{\text{inf}}/\xi A_{\text{inf}}$ and obtain a surjective homomorphism

$$A_{\text{inf}} \twoheadrightarrow \varprojlim_n A_{\text{inf}}/(p^n A_{\text{inf}} + \xi A_{\text{inf}}) \cong \varprojlim_n (A_{\text{inf}}/\xi A_{\text{inf}})/((p^n A_{\text{inf}} + \xi A_{\text{inf}})/\xi A_{\text{inf}}) = \widehat{A_{\text{inf}}/\xi A_{\text{inf}}}$$

as A_{inf} is p-adically complete. The kernel of this map is $\displaystyle\bigcap_{n=1}^{\infty} (p^n A_{\text{inf}} + \xi A_{\text{inf}})$, which clearly contains ξA_{inf}. Therefore it suffices to show that every $c \in \displaystyle\bigcap_{n=1}^{\infty} (p^n A_{\text{inf}} + \xi A_{\text{inf}})$ lies in ξA_{inf}. Take sequences (a_n) and (b_n) in A_{inf} with $c = p^n a_n + \xi b_n$ for each $n \geq 1$. We have

$$p^n(a_n - pa_{n+1}) = \xi(b_{n+1} - b_n) \quad \text{for each } n \geq 1.$$

Since ξ has a nonzero image in $A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$, each $b_{n+1} - b_n$ is divisible by p^n in A_{inf}. Now we see that the sequence (b_n) converges to an element $b \in A_{\text{inf}}$ and in turn find

$$c = \lim_{n\to\infty} (p^n a_n + \xi b_n) = \lim_{n\to\infty} p^n a_n + \xi \lim_{n\to\infty} b_n = \xi b,$$

thereby completing the proof. □

Definition 1.1.12. Given a primitive element $\xi \in A_{\text{inf}}$, we refer to the natural projection $\theta_\xi : A_{\text{inf}} \twoheadrightarrow A_{\text{inf}}/\xi A_{\text{inf}}$ as the *untilt map* associated to ξ.

LEMMA 1.1.13. Let ξ be a strongly primitive element in A_{inf}.

(1) For every nonzero $c \in \mathcal{O}_F$, some power of p is divisible by $\theta_\xi([c])$ in $A_{\text{inf}}/\xi A_{\text{inf}}$.
(2) For every $m \in \mathfrak{m}_F$, some power of $\theta_\xi([m])$ is divisible by p in $A_{\text{inf}}/\xi A_{\text{inf}}$.

PROOF. Let us write $\xi = [\varpi] - pu$ for some $\varpi \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$. Every nonzero $c \in \mathcal{O}_F$ gives rise to an expression $\varpi^i = cc'$ for some $i > 0$ and $c' \in \mathcal{O}_F$, thereby yielding an equality

$$p^i = \big(\theta_\xi(u^{-1})\theta_\xi(up)\big)^i = \theta_\xi(u^{-1})^i \theta_\xi([\varpi])^i = \theta_\xi(u^{-1})^i \theta_\xi([c])^i \theta_\xi([c'])^i.$$

Similarly, every $m \in \mathfrak{m}_F$ admits an expression $m^j = \varpi b$ for some $j > 0$ and $b \in \mathcal{O}_F$, thereby yielding an equality

$$\theta_\xi([m])^j = \theta_\xi([\varpi])\theta_\xi([b]) = \theta_\xi(pu)\theta_\xi([b]) = p\theta_\xi(u)\theta_\xi([b]).$$

Hence we establish the desired assertions. □

Proposition 1.1.14. Let ξ be a strongly primitive element in A_{inf}. An element $c \in \mathcal{O}_F$ divides another element $c' \in \mathcal{O}_F$ if and only if $\theta_\xi([c])$ divides $\theta_\xi([c'])$ in $A_{\text{inf}}/\xi A_{\text{inf}}$.

Proof. If c divides c' in $\mathcal{O}_F$, we see that $\theta_\xi([c])$ divides $\theta_\xi([c'])$ in $A_{\text{inf}}/\xi A_{\text{inf}}$ as the Teichmüller lifts are multiplicative. Let us now assume that c does not divide c' in $\mathcal{O}_F$. We wish to show that $\theta_\xi([c])$ does not divide $\theta_\xi([c'])$ in $A_{\text{inf}}/\xi A_{\text{inf}}$. Suppose for contradiction that there exists an element $a \in A_{\text{inf}}/\xi A_{\text{inf}}$ with $\theta_\xi([c']) = \theta_\xi([c])a$. Since we have $\nu_F(c) > \nu_F(c')$, there exists some $m \in \mathfrak{m}_F$ with $c = mc'$. Hence we find

$$\theta_\xi([c']) = \theta_\xi([c])a = \theta_\xi([c'])\theta_\xi([m])a.$$

Meanwhile, Proposition 1.1.11 and Lemma 1.1.13 together imply that $\theta_\xi([c'])$ is not a zero divisor in $A_{\text{inf}}/\xi A_{\text{inf}}$. Now we obtain the equality $\theta_\xi([m])a = 1$, which yields a desired contradiction as the image of $\theta_\xi([m])$ under the natural map $A_{\text{inf}}/\xi A_{\text{inf}} \twoheadrightarrow A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}})$ is nilpotent by Lemma 1.1.13. □

Proposition 1.1.15. Let ξ be a strongly primitive element in A_{inf}. Every $a \in A_{\text{inf}}/\xi A_{\text{inf}}$ is a unit multiple of $\theta_\xi([c])$ for some $c \in \mathcal{O}_F$ which is unique up to unit multiple.

Proof. If a is a unit multiple of $\theta_\xi([c_1])$ and $\theta_\xi([c_2])$ for some $c_1, c_2 \in \mathcal{O}_F$, we see that $\theta_\xi([c_1])$ and $\theta_\xi([c_2])$ are unit multiples of each other, which means by Proposition 1.1.14 that c_1 and c_2 are unit multiples of each other. Hence it remains to prove that a is a unit multiple of $\theta_\xi([c])$ for some $c \in \mathcal{O}_F$. Since the assertion is evident for $a = 0$, we henceforth assume that a is nonzero. By Proposition 1.1.11, we may write $a = p^n a'$ for some $n \geq 0$ and $a' \in A_{\text{inf}}/\xi A_{\text{inf}}$ such that a' is not divisible by p. Let us take $\varpi \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$ with $\xi = [\varpi] - up$. We find that a admits the identity

$$a = p^n a' = \left(\theta_\xi(u^{-1})\theta_\xi(up)\right)^n a' = \theta_\xi(u)^{-1}\theta_\xi([\varpi])^n a'.$$

Hence we may replace a by a' to assume that a is not divisible by p.

We observe that there exists a natural isomorphism

$$A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}}) = A_{\text{inf}}/([\varpi]A_{\text{inf}} + pA_{\text{inf}}) \cong \mathcal{O}_F/\varpi\mathcal{O}_F$$

and in turn obtain a commutative diagram

$$\begin{array}{ccc} A_{\text{inf}} & \xrightarrow{\ \theta_\xi\ } & A_{\text{inf}}/\xi A_{\text{inf}} \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}_F \cong A_{\text{inf}}/pA_{\text{inf}} & \twoheadrightarrow & A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}}) \cong \mathcal{O}_F/\varpi\mathcal{O}_F \end{array} \tag{1.1}$$

where all arrows are evidently surjective. Choose an element $c \in \mathcal{O}_F$ whose image under the bottom horizontal arrow coincides with the image of a under the second vertical arrow. We see that c is not divisible by ϖ as a is not divisible by p. Hence we write $\varpi = cm$ for some $m \in \mathfrak{m}_F$ and find

$$p = \theta_\xi(u^{-1})\theta_\xi(up) = \theta_\xi(u)^{-1}\theta_\xi([\varpi]) = \theta_\xi(u)^{-1}\theta_\xi([c])\theta_\xi([m]).$$

Moreover, the diagram (1.1) yields an element $b \in A_{\text{inf}}/\xi A_{\text{inf}}$ with

$$a = \theta_\xi([c]) + pb = \theta_\xi([c]) + b\theta_\xi(u)^{-1}\theta_\xi([c])\theta_\xi([m]) = \theta_\xi([c])\left(1 + b\theta_\xi(u)^{-1}\theta_\xi([m])\right).$$

Now we complete the proof by noting that $1 + b\theta_\xi(u)^{-1}\theta_\xi([m])$ is a unit in $A_{\text{inf}}/\xi A_{\text{inf}}$ with

$$\left(1 + b\theta_\xi(u)^{-1}\theta_\xi([m])\right)^{-1} = \sum_{i=0}^{\infty} (-1)^i \left(b\theta_\xi(u)^{-1}\theta_\xi([m])\right)^i$$

where the convergence of the sum follows from Proposition 1.1.11 and Lemma 1.1.13. □

PROPOSITION 1.1.16. Let ξ be a primitive element in A_{inf}.

(1) The ring $A_{\text{inf}}/\xi A_{\text{inf}}$ is an integral domain, whose fraction field C_ξ is naturally an untilt of F with $\mathcal{O}_{C_\xi} = A_{\text{inf}}/\xi A_{\text{inf}}$.
(2) Each element $c \in \mathcal{O}_F$ maps to $\theta_\xi([c])$ under the sharp map associated to C_ξ.

PROOF. Let us write $\xi = [\varpi] - up$ with $\varpi \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$. In addition, we let $\mathcal{O}$ denote the ring $A_{\text{inf}}/\xi A_{\text{inf}}$. If ϖ is zero, we have $\mathcal{O} = A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$ and thus establish the desired assertions by Example 1.1.20. We henceforth assume that ϖ is nonzero.

For every nonzero $c \in \mathcal{O}_F$, Proposition 1.1.11 and Lemma 1.1.13 and together show that $\theta_\xi([c])$ is not a zero divisor. Hence Proposition 1.1.15 implies that $\mathcal{O}$ is an integral domain. In addition, Proposition 1.1.15 yields a function $\nu^+ : \mathcal{O} \to [0, \infty]$ with

$$\nu^+(w\theta_\xi([c])) = \nu_F(c) \quad \text{for every } c \in \mathcal{O}_F \text{ and } w \in \mathcal{O}^\times.$$

It is evident that ν^+ is a monoid homomorphism with respect to the multiplication on $\mathcal{O}$ and the addition on $[0, \infty]$. The image of ν^+ is nondiscrete as the valuation ν_F is nondiscrete. Moreover, for arbitrary nonzero $a, b \in \mathcal{O}$ with $\nu^+(a) \geq \nu^+(b)$, we see by Proposition 1.1.14 that a is divisible b in $\mathcal{O}$ and in turn find

$$\nu^+(a + b) = \nu^+((a/b + 1)b) = \nu^+(a/b + 1) + \nu^+(b) \geq \nu^+(b) = \min(\nu^+(a), \nu^+(b)).$$

Therefore ν^+ naturally extends to a nondiscrete valuation ν on the fraction field C_ξ of $\mathcal{O}$. Proposition 1.1.14 implies that the valuation ring of C_ξ is $\mathcal{O}$; indeed, for every $x = a/b \in C_\xi$ with $a, b \in \mathcal{O}$, its valuation $\nu(a) - \nu(b)$ is nonnegative if and only if a is divisible by b in $\mathcal{O}$. In addition, since we have

$$\nu(p) = \nu\left(\theta_\xi(u^{-1})\theta_\xi(up)\right) = \nu\left(\theta_\xi(u)^{-1}\theta_\xi([\varpi])\right) = \nu_F(\varpi) > 0,$$

we see that C_ξ has residue characteristic p and also find by Proposition 1.1.11 that C_ξ is complete with respect to the valuation ν. Meanwhile, the surjectivity of the p-th power map on $\mathcal{O}_F/\varpi\mathcal{O}_F$ yields the surjectivity of the p-th power map on $\mathcal{O}_{C_\xi}/p\mathcal{O}_{C_\xi}$ via the natural isomorphism

$$\mathcal{O}_F/\varpi\mathcal{O}_F \cong A_{\text{inf}}/([\varpi]A_{\text{inf}} + pA_{\text{inf}}) = A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}}) \cong \mathcal{O}_{C_\xi}/pC_\xi. \tag{1.2}$$

Therefore C_ξ is a perfectoid field with $\mathcal{O}_{C_\xi} = A_{\text{inf}}/\xi A_{\text{inf}}$.

By Proposition 2.1.7 in Chapter III, the isomorphism (1.2) gives rise to a topological isomorphism

$$\mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_F/\varpi\mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_{C_\xi}/p\mathcal{O}_{C_\xi} \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_{C_\xi} \cong \mathcal{O}_{C_\xi^\flat}$$

which uniquely extends to a topological isomorphism $\iota : F \simeq C_\xi^\flat$. It is not hard to see that each $c \in \mathcal{O}_F$ maps to $(\theta_\xi([c^{1/p^n}]) \in \mathcal{O}_{C_\xi^\flat}$ under ι and in turn maps to $\theta_\xi([c])$ under the sharp map associated to C_ξ. Therefore we have established the desired assertions. □

Remark. It is evident by our proof that the valuation ν on C_ξ coincides with the normalized valuation on C_ξ.

Definition 1.1.17. For every primitive element $\xi \in A_{\text{inf}}$, we refer to the untilt C_ξ of F constructed in Proposition 1.1.16 as the *untilt of F associated to ξ*.

PROPOSITION 1.1.18. For a primitive element $\xi \in A_{\text{inf}}$, the untilt C_ξ of F has characteristic 0 if and only if ξ is strongly primitive.

PROOF. The assertion is straightforward to verify by Proposition 1.1.11. □

Definition 1.1.19. Two untilts C_1 and C_2 of F are *equivalent* if there exists a topological isomorphism $f : C_1 \simeq C_2$ with a commutative diagram

$$\begin{array}{ccc} C_1^\flat & \xrightarrow[f^\flat]{\sim} & C_2^\flat \\ & {\scriptstyle \iota_{C_1}} \nwarrow \sim \quad \sim \nearrow {\scriptstyle \iota_{C_2}} & \\ & F & \end{array}$$

where $f^\flat$ is the isomorphism induced by f.

Example 1.1.20. Proposition 2.1.14 in Chapter III implies that every untilt of F in characteristic p is equivalent to the trivial untilt of F.

Proposition 1.1.21. Given a perfectoid field C, every topological isomorphism $\iota : F \simeq C^\flat$ naturally induces an isomorphism $\overline{\iota} : \mathcal{O}_F/\varpi\mathcal{O}_F \simeq \mathcal{O}_C/p\mathcal{O}_C$ for some $\varpi \in \mathcal{O}_F$.

Proof. Let us regard C as an untilt of F via the topological isomorphism $\iota : F \simeq C^\flat$ and write ν for the normalized valuation on C. We apply Proposition 2.1.11 in Chapter III to find an element $\varpi \in \mathcal{O}_F$ with $\nu_F(\varpi) = \nu(p)$. In addition, we note that ι restricts to an isomorphism $\mathcal{O}_F \simeq \mathcal{O}_{C^\flat}$ and in turn induces a map

$$\mathcal{O}_F \xrightarrow{c \mapsto c^\sharp} \mathcal{O}_C \twoheadrightarrow \mathcal{O}_C/p\mathcal{O}_C.$$

This map is a surjective ring homomorphism by Proposition 2.1.10 in Chapter III. Its kernel consists of the elements $c \in \mathcal{O}_F$ with $\nu(c^\sharp) \geq \nu(p)$ or equivalently $\nu_F(c) \geq \nu_F(\varpi)$. Therefore we obtain an isomorphism $\overline{\iota} : \mathcal{O}_F/\varpi\mathcal{O}_F \simeq \mathcal{O}_C/p\mathcal{O}_C$ induced by ι as desired. $\square$

Remark. It is evident from our proof that the ring $\mathcal{O}_F/\varpi\mathcal{O}_F$ and the isomorphism $\overline{\iota}$ do not depend on the choice of the element $\varpi \in \mathfrak{m}_F$ with $\nu_F(\varpi) = \nu(p)$. Moreover, we can show that $\mathcal{O}_F/\varpi\mathcal{O}_F$ depends only on the perfectoid field C.

Definition 1.1.22. Given a perfectoid field C and a topological isomorphism $\iota : F \simeq C^\flat$, we refer to the map $\overline{\iota}$ in Proposition 1.1.21 as the *sharp map reduction* of ι.

Example 1.1.23. Let ξ be a primitive element in A_{inf}. If we write ϖ for the image of ξ in $A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$, we apply Proposition 1.1.16 to identify $\overline{\iota_{C_\xi}}$ with the natural isomorphism

$$\mathcal{O}_F/\varpi\mathcal{O}_F \cong A_{\text{inf}}/([\varpi]A_{\text{inf}} + pA_{\text{inf}}) = A_{\text{inf}}/(\xi A_{\text{inf}} + pA_{\text{inf}}) \cong \mathcal{O}_{C_\xi}/pC_\xi.$$

Hence for each $c \in \mathcal{O}_F$, the isomorphism $\overline{\iota_{C_\xi}}$ maps the image of c in $\mathcal{O}_F/\varpi\mathcal{O}_F$ to the image of $\theta_\xi([c])$ in $\mathcal{O}_{C_\xi}/pC_\xi$.

Proposition 1.1.24. Given a perfectoid field C, two topological isomorphisms $\iota_1 : F \simeq C^\flat$ and $\iota_2 : F \simeq C^\flat$ coincide if and only if their sharp map reductions coincide.

Proof. If ι_1 and ι_2 coincide, their sharp map reductions evidently coincide. Let us now assume for the converse that $\overline{\iota}_1$ and $\overline{\iota}_2$ are equal. Choose an element $\varpi \in \mathcal{O}_F$ such that $\mathcal{O}_F/\varpi\mathcal{O}_F$ is the source for $\overline{\iota}_1 = \overline{\iota}_2$. By Proposition 2.1.7 in Chapter III, the map $\overline{\iota}_1 = \overline{\iota}_2$ induces a topological isomorphism

$$\mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_F/\varpi\mathcal{O}_F \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C/p\mathcal{O}_C \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_C \cong \mathcal{O}_{C^\flat}$$

and in turn yields a topological isomorphism $\iota : F \simeq C^\flat$. Hence we see that ι coincides with both ι_1 and ι_2 by construction, thereby completing the proof. $\square$

Remark. Proposition 1.1.24 shows that we can recover a topological isomorphism $\iota : F \simeq C^\flat$ from its sharp map reduction $\overline{\iota}$, even though $\overline{\iota}$ is purely algebraic.

PROPOSITION 1.1.25. Let C be an untilt of F.

(1) There exists a surjective ring homomorphism $\theta_C : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$ with

$$\theta_C\left(\sum [c_n]p^n\right) = \sum c_n^\sharp p^n \quad \text{for every } c_n \in \mathcal{O}_F. \tag{1.3}$$

(2) The ideal $\ker(\theta_C)$ of A_{inf} contains a primitive element.

(3) Every primitive element in $\ker(\theta_C)$ generates $\ker(\theta_C)$.

PROOF. Given a p-adic field K, all results from the first part of §2.2 in Chapter III rely only on the fact that $\mathbb{C}_K$ is an algebraically closed perfectoid field. Since C is algebraically closed as noted in Proposition 1.1.6, these results remain valid with C in place of $\mathbb{C}_K$. In particular, we obtain a surjective ring homomorphism $\theta_C : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$ with the equality (1.3) by Proposition 2.2.4 in Chapter III and find a primitive element $\xi_C \in A_{\text{inf}}$ generating $\ker(\theta_C)$ by Proposition 2.2.12 in Chapter III.

It remains to prove that every primitive element $\xi \in \ker(\theta_C)$ generates $\ker(\theta_C)$. Let us choose a valuation ν on $\mathcal{O}_{C_\xi}$ and write $\overline{\xi_C}$ for the image of ξ_C in $A_{\text{inf}}/\xi A_{\text{inf}} = \mathcal{O}_{C_\xi}$. The surjective ring homomorphism θ_C gives rise to an isomorphism

$$\mathcal{O}_{C_\xi}/\overline{\xi_C}\mathcal{O}_{C_\xi} \cong A_{\text{inf}}/\xi_C A_{\text{inf}} = A_{\text{inf}}/\ker(\theta_C) \simeq \mathcal{O}_C.$$

If $\overline{\xi_C}$ is nonzero, we see that every $a \in \mathcal{O}_{C_\xi}$ with $0 < \nu(a) < \nu(\overline{\xi_C})$ yields a nilpotent element in $\mathcal{O}_{C_\xi}/\overline{\xi_C}\mathcal{O}_{C_\xi} \simeq \mathcal{O}_C$, which is impossible. Therefore we find $\overline{\xi_C} = 0$ and in turn deduce that ξ generates $\ker(\theta_C) = \xi_C A_{\text{inf}}$ as asserted in statement (3). □

Remark. It is not hard to show that every generator of $\ker(\theta_C)$ is a primitive element.

Definition 1.1.26. Given an untilt C of F, we refer to the map θ_C in Proposition 1.1.25 as the *Fontaine map* of C and let $\theta_C[1/p] : A_{\text{inf}}[1/p] \to C$ denote the induced ring homomorphism.

THEOREM 1.1.27 (Kedlaya-Liu [**KL15**], Fontaine [**Fon13**]). There is a natural bijection

$$\{ \text{ equivalence classes of untilts of } F \} \xrightarrow{\sim} \{ \text{ ideals of } A_{\text{inf}} \text{ generated by a primitive element } \}$$

which maps each untilt C of F to $\ker(\theta_C)$.

PROOF. Take an arbitrary primitive element $\xi \in A_{\text{inf}}$. By Proposition 1.1.16, each $c \in \mathcal{O}_F$ maps to $\theta_\xi([c])$ under the sharp map associated to C_ξ. We see that θ_ξ coincides with θ_{C_ξ} and consequently find $\xi A_{\text{inf}} = \ker(\theta_\xi) = \ker(\theta_{C_\xi})$.

It remains to prove that every untilt C of F with $\ker(\theta_C) = \xi A_{\text{inf}}$ is equivalent to C_ξ. Proposition 1.1.25 shows that θ_C yields a topological isomorphism $\tilde{f} : \mathcal{O}_{C_\xi} = A_{\text{inf}}/\xi A_{\text{inf}} \simeq \mathcal{O}_C$ with $\tilde{f}(\theta_\xi([c])) = c^\sharp$ for each $c \in \mathcal{O}_F$. Moreover, the map $\tilde{f}$ uniquely extends to a topological isomorphism $f : C_\xi \simeq C$ and gives rise to an isomorphism $\overline{f} : \mathcal{O}_{C_\xi}/p\mathcal{O}_{C_\xi} \simeq \mathcal{O}_C/p\mathcal{O}_C$. Let us take the topological isomorphism $f^\flat : C_\xi^\flat \simeq C^\flat$ induced by f and denote by ϖ the image of ξ in $A_{\text{inf}}/pA_{\text{inf}} \cong \mathcal{O}_F$. For each $c \in \mathcal{O}_F$, we apply Example 1.1.23 to see that $\overline{f} \circ \overline{\iota_{C_\xi}}$ maps the image of c in $\mathcal{O}_F/\varpi\mathcal{O}_F$ to the image of $\tilde{f}(\theta_\xi([c])) = c^\sharp$. Hence we find

$$\overline{\iota_C} = \overline{f} \circ \overline{\iota_{C_\xi}} = \overline{f^\flat \circ \iota_{C_\xi}}$$

and in turn obtain the equality $\iota_C = f^\flat \circ \iota_{C_\xi}$ by Proposition 1.1.24, thereby establishing the desired assertion. □

1.2. The algebraic Fargues-Fontaine curve

The main objective for this subsection is to construct the Fargues-Fontaine curve as a scheme. For the rest of this chapter, we fix a nonzero element $\varpi \in \mathfrak{m}_F$ and denote by $Y = Y_F$ the set of equivalence classes of untilts of F in characteristic 0.

Definition 1.2.1. Let C be an untilt of F and x be an element of C.

(1) We define the *normalized absolute value* of x to be $|x|_C := p^{-\nu_C(x)}$.
(2) For $C = F$, we often refer to $|x| := |x|_F$ simply as the *absolute value* of x.

Remark. For the rest of this chapter, we will often use absolute values instead of valuations for notational convenience, especially in arguments which involve analytic methods.

Example 1.2.2. Given an untilt C of F, Theorem 1.1.27 yields a primitive element $\xi \in A_{\text{inf}}$ which generates $\ker(\theta_C)$. If we write $\xi = [m] - up$ for some $m \in \mathfrak{m}_F$ and $u \in A_{\text{inf}}^\times$, we have

$$|p|_C = \left|\theta_C(u)^{-1}\theta_C([m])\right|_C = |\theta_C([m])|_C = \left|m^\sharp\right|_C = |m|.$$

PROPOSITION 1.2.3. The ring $A_{\text{inf}}[1/p, 1/[\varpi]]$ admits an identification

$$A_{\text{inf}}[1/p, 1/[\varpi]] = \left\{ \sum [c_n]p^n \in W(F)[1/p] : |c_n| \text{ bounded} \right\}.$$

PROOF. Given an element $f = \sum [c_n]p^n \in W(F)[1/p]$, we have $f \in A_{\text{inf}}[1/p, 1/[\varpi]]$ if and only if there exists an integer $i \geq 0$ with $[\varpi^i]f = \sum [c_n\varpi^i]p^n \in A_{\text{inf}}[1/p]$ or equivalently $|c_n| \leq \left|\varpi^{-i}\right|$ for each $n \in \mathbb{Z}$. $\square$

Remark. Proposition 1.2.3 shows that the ring $A_{\text{inf}}[1/p, 1/[\varpi]]$ does not depend on our choice of the nonzero element $\varpi \in \mathfrak{m}_F$.

LEMMA 1.2.4. Given two equivalent untilts C and C' of F, we have $|p|_C = |p|_{C'}$.

PROOF. Since $\ker(\theta_C)$ and $\ker(\theta_{C'})$ coincide by Theorem 1.1.27, the desired assertion follows from Example 1.2.2. $\square$

Definition 1.2.5. Let y be an element of Y and C be a representative of y.

(1) The *absolute value* of y is $|y| := |p|_C$.
(2) The *extended Fontaine map* of C is the ring homomorphism $\widetilde{\theta_C} : A_{\text{inf}}[1/p, 1/[\varpi]] \to C$ which extends the Fontaine map $\theta_C : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$.
(3) Given an element $f \in A_{\text{inf}}[1/p, 1/[\varpi]]$, its *$C$-value at y* is $f(y)_C := \widetilde{\theta_C}(f)$, often denoted by $f(y)$ if the context clearly specifies C.

Remark. In order to understand the structures of the Fargues-Fontaine curve, it is often useful to regard Y as an analogue of the punctured unit disk $\mathbb{D}^* := \{ z \in \mathbb{C} : 0 < |z| < 1 \}$ in the complex plane. Here we present a couple of similarities between Y and $\mathbb{D}^*$.

(1) For each $y \in Y$ represented by an untilt C of F, its absolute value $|y| = |p|_C$ is a real number between 0 and 1. Similarly, for every $z \in \mathbb{D}^*$ its absolute value $|z|$ is a real number between 0 and 1.
(2) Every element in $A_{\text{inf}}[1/p, 1/[\varpi]]$ is a "Laurent series in the variable p" with bounded coefficients and gives rise to a function on Y as described in Definition 1.2.5. Similarly, every Laurent series $\sum a_n z^n$ over $\mathbb{C}$ with bounded coefficients defines a holomorphic function on $\mathbb{D}^*$.

LEMMA 1.2.6. Let ρ be a real number with $0 < \rho < 1$ and f be an element in $A_{\text{inf}}[1/p, 1/[\varpi]]$ with a Teichmüller expansion $f = \sum [c_n]p^n$.

(1) The sequence $(|c_n| \rho^n)$ is bounded.
(2) There exist finitely many integers m with $\sup_{n\in\mathbb{Z}}(|c_n| \rho^n) = |c_m| \rho^m$

PROOF. Let us take an integer n_0 with $c_{n_0} \neq 0$. Proposition 1.2.3 yields an integer $l > 0$ with $|c_n| \rho^n < |c_{n_0}| \rho^{n_0}$ for each $n > l$. Moreover, there exists an integer $m < 0$ with $c_n = 0$ for each $n < m$. Hence the sequence $(|c_n| \rho^n)$ is bounded with $\sup_{n\in\mathbb{Z}}(|c_n| \rho^n) = \sup_{m<n<l}(|c_n| \rho^n)$. □

Definition 1.2.7. Let ρ be a real number with $0 < \rho < 1$ and f be an element in $A_{\text{inf}}[1/p, 1/[\varpi]]$ with a Teichmüller expansion $f = \sum [c_n]p^n$.

(1) We define the *Gauss ρ-norm* of f to be $|f|_\rho := \sup_{n\in\mathbb{Z}}(|c_n| \rho^n)$.
(2) We say that ρ is *generic* for f if there exists a unique integer n with $|f|_\rho = |c_n| \rho^n$.

LEMMA 1.2.8. Given an element $f \in A_{\text{inf}}[1/p, 1/[\varpi]]$, the set

$$S_f := \{ \rho \in (0, 1) : \rho \text{ is generic for } f \}$$

is dense in the interval $(0, 1)$.

PROOF. Let us write $f = \sum [c_n]p^n$ with $c_n \in F$. If $\rho \in (0, 1)$ is not generic for f, we find $|f|_\rho = |c_m| \rho^m = |c_n| \rho^n$ for some distinct $m, n \in \mathbb{Z}$ by Lemma 1.2.6 and in turn obtain the equality $\rho = (|c_m| / |c_n|)^{1/(n-m)}$. Hence we deduce that the complement of S_f in $(0, 1)$ is countable, thereby establishing the desired assertion. □

LEMMA 1.2.9. Let y be an element of Y and C be a representative of y. Given an element $f \in A_{\text{inf}}[1/p, 1/[\varpi]]$, we have $|f(y)|_C \leq |f|_{|y|}$ with equality if $|y|$ is generic for f.

PROOF. We write $f = \sum [c_n]p^n$ with $c_n \in F$ and find

$$|f(y)|_C = \left| \sum c_n^\sharp p^n \right|_C \leq \sup_{n\in\mathbb{Z}} \left(\left| c_n^\sharp \right|_C |p|_C^n \right) = \sup_{n\in\mathbb{Z}} (|c_n| \, |y|^n) = |f|_{|y|} \, .$$

It is evident that the inequality becomes an equality if $|y|$ is generic for f. □

PROPOSITION 1.2.10. For every $\rho \in (0, 1)$, the Gauss ρ-norm on $A_{\text{inf}}[1/p, 1/[\varpi]]$ is a multiplicative nonarchimedean norm.

PROOF. Let f and g be arbitrary elements in $A_{\text{inf}}[1/p, 1/[\varpi]]$. We note that the value group $|F|$ of F is dense in $[0, \infty)$ and in turn apply Lemma 1.2.8 to see that the set

$$S := \{ \tau \in (0, 1) \cap |F| : \tau \text{ is generic for } f, g, f + g, \text{ and } fg \}$$

is dense in $(0, 1)$. Let us write $\rho = \lim_{n\to\infty} \tau_n$ with $\tau_n \in S$ and choose a sequence (m_n) in $\mathfrak{m}_F$ with $|m_n| = \tau_n$. Since each $\xi_n := [m_n] - p \in A_{\text{inf}}$ is strongly primitive, Theorem 1.1.27 and Example 1.2.2 together yield a sequence (y_n) in Y with $|y_n| = \tau_n$. For each $n \geq 0$, we take a representative C_n of y_n and use Lemma 1.2.9 to find

$$|f + g|_{\tau_n} = |f(y_n) + g(y_n)|_{C_n} \leq \max(|f(y_n)|_{C_n}, |g(y_n)|_{C_n}) = \max(|f|_{\tau_n}, |g|_{\tau_n}),$$
$$|fg|_{\tau_n} = |f(y_n)g(y_n)|_{C_n} = |f(y_n)|_{C_n} |g(y_n)|_{C_n} = |f|_{\tau_n} |g|_{\tau_n} \, .$$

Hence we take limits to obtain the relations

$$|f + g|_\rho \leq \max(|f|_\rho, |g|_\rho) \quad \text{and} \quad |fg|_\rho = |f|_\rho |g|_\rho \, .$$

thereby completing the proof. □

Definition 1.2.11. Let $[a, b]$ be a closed subinterval of $(0, 1)$.

(1) The $[a, b]$-*annulus of untilts* is

$$Y_{[a,b]} := \{ \, y \in Y : a \leq |y| \leq b \, \} .$$

(2) The *ring of holomorphic functions on* $Y_{[a,b]}$, denoted by $B_{[a,b]}$, is the completion of $A_{\inf}[1/p, 1/[\varpi]]$ with respect to the Gauss a-norm and the Gauss b-norm.

Lemma 1.2.12. Given a closed subinterval $[a, b]$ of $(0, 1)$, every $f \in A_{\inf}[1/p, 1/[\varpi]]$ satisfies the inequality $|f|_\rho \leq \sup(|f|_a, |f|_b)$ for each $\rho \in [a, b]$.

Proof. If we write $f = \sum [c_n] p^n$ with $c_n \in F$, we find

$$\begin{aligned} |c_n| \, \rho^n &\leq |c_n| \, b^n \leq |f|_b \qquad \text{for each } n \geq 0, \\ |c_n| \, \rho^n &\leq |c_n| \, a^n \leq |f|_a \qquad \text{for each } n < 0. \end{aligned}$$

Hence we obtain the desired assertion. □

Remark. Since the value group $|F|$ of F is dense in $(0, \infty)$, we find

$$\sup_{|y|=\rho} (|f(y)|_C) = |f|_\rho \qquad \text{for each } \rho \in |F| \cap (0, 1)$$

by Lemma 1.2.8 and Lemma 1.2.9. Hence we may regard Lemma 1.2.12 as an analogue of the maximum modulus principle for holomorphic functions on $\mathbb{D}^*$.

Proposition 1.2.13. Let $[a, b]$ be a closed subinterval of $(0, 1)$.

(1) The ring $B_{[a,b]}$ is the completion of $A_{\inf}[1/p, 1/[\varpi]]$ with respect to all Gauss ρ-norms with $\rho \in [a, b]$.

(2) For every closed interval $[a', b']$ with $[a, b] \subseteq [a', b'] \subseteq (0, 1)$, there exists a natural ring homomorphism $B_{[a',b']} \to B_{[a,b]}$.

Proof. Statement (1) is evident by Lemma 1.2.12. Statement (2) is an immediate consequence of statement (1). □

Definition 1.2.14. We define the *ring of holomorphic functions on* $Y = Y_F$ to be

$$B = B_F := \varprojlim B_{[a,b]}$$

where the transition maps are the natural homomorphisms given by Proposition 1.2.13.

Remark. A sum $\sum [c_n] p^n$ with $c_n \in F$ converges in B if and only if it satisfies the relations

$$\limsup_{n>0} |c_n|^{1/n} \leq 1 \quad \text{ and } \quad \lim_{n \to \infty} |c_{-n}|^{1/n} = 0.$$

Similarly, a Laurent series $\sum a_n z^n$ over $\mathbb{C}$ converges on $\mathbb{D}^*$ if and only if it satisfies the relations

$$\limsup_{n>0} |a_n|^{1/n} \leq 1 \quad \text{ and } \quad \lim_{n \to \infty} |a_{-n}|^{1/n} = 0.$$

However, an arbitrary element in B does not necessarily admit a "Laurent series expansion", whereas every holomorphic function on $\mathbb{D}^*$ admits a unique Laurent series expansion.

Lemma 1.2.15. Let $\eta : R_1 \to R_2$ be a continuous homomorphism of normed rings.

(1) The map η uniquely extends to a continuous ring homomorphism $\widehat{\eta} : \widehat{R_1} \to \widehat{R_2}$ where $\widehat{R_1}$ and $\widehat{R_2}$ respectively denote the completions of R_1 and R_2.

(2) If η is a homeomorphism, $\widehat{\eta}$ is also a homeomorphism.

Proof. The assertions are straightforward to verify. □

PROPOSITION 1.2.16. Let C be an untilt of F in characteristic 0. The map θ_C uniquely extends to a surjective continuous open ring homomorphism $\widehat{\theta_C} : B \twoheadrightarrow C$.

PROOF. Lemma 1.2.9 and Lemma 1.2.15 show that $\widetilde{\theta_C}$ uniquely extends to a continuous ring homomorphism $\widehat{\widetilde{\theta_C}} : B_{[\rho,\rho]} \to C$ with $\rho := |p|_C$. Take $\widehat{\theta_C}$ to be the composition of $\widehat{\widetilde{\theta_C}}$ with the natural map $B \to B_{[\rho,\rho]}$. It is evident that $\widehat{\theta_C}$ is surjective and continuous. Moreover, the open mapping theorem implies that $\widehat{\theta_C}$ is open. Hence we establish the desired assertion. □

Definition 1.2.17. Let y be an element in Y and C be a representative of y.

(1) The *completed Fontaine map* of C is the ring homomorphism $\widehat{\theta_C} : B \twoheadrightarrow C$ constructed in Proposition 1.2.16.
(2) For every $f \in B$, its *C-value at y* is $f(y)_C := \widehat{\theta_C}(f)$, often denoted by $f(y)$ if the context clearly specifies C.

PROPOSITION 1.2.18. The Frobenius automorphism of A_{inf} uniquely extends to topological isomorphisms $\varphi : B \simeq B$ and $\varphi_{[a,b]} : B_{[a,b]} \simeq B_{[a^p,b^p]}$ for every closed interval $[a,b] \subseteq (0,1)$.

PROOF. The Frobenius automorphism on $A_{\text{inf}} = W(\mathcal{O}_F)$ extends to the Frobenius automorphism on $W(F)$. Since we have

$$\varphi_{W(F)}\left(\sum [c_n]p^n\right) = \sum [c_n^p]p^n \qquad \text{for each } c_n \in F, \tag{1.4}$$

Proposition 1.2.3 implies that $\varphi_{W(F)}$ restricts to an automorphism on $A_{\text{inf}}[1/p, 1/[\varpi]]$, which we denote by φ_{inf}. By the identity (1.4), we find

$$|\varphi_{\text{inf}}(f)|_{\rho^p} = |f|_\rho^p \qquad \text{for every } f \in A_{\text{inf}}[1/p, 1/[\varpi]] \text{ and } \rho \in (0,1). \tag{1.5}$$

Consider an arbitrary closed interval $[a,b] \subseteq (0,1)$ and choose a real number $r \in [a,b]$. Lemma 1.2.15 and the identity (1.5) show that the automorphism φ_{inf} on $A_{\text{inf}}[1/p, 1/[\varpi]]$ uniquely extends to a topological isomorphism $\varphi_{[r,r]} : B_{[r,r]} \simeq B_{[r^p,r^p]}$. Moreover, by the identity (1.5) a sequence (f_n) in $A_{\text{inf}}[1/p, 1/[\varpi]]$ is Cauchy with respect to the Gauss a-norm and the Gauss b-norm if and only if the sequence $(\varphi_{\text{inf}}(f_n))$ in $A_{\text{inf}}[1/p, 1/[\varpi]]$ is Cauchy with respect to the Gauss a^p-norm and the Gauss b^p-norm. Hence we deduce that $\varphi_{[r,r]}$ restricts to a topological isomorphism $\varphi_{[a,b]} : B_{[a,b]} \simeq B_{[a^p,b^p]}$ with an inverse given by the restriction of $\varphi_{[r,r]}^{-1}$ on $B_{[a^p,b^p]}$. It is evident by construction that $\varphi_{[a,b]}$ is an extension of φ_{inf}.

By our discussion in the preceding paragraph, the automorphism φ_{inf} on $A_{\text{inf}}[1/p, 1/[\varpi]]$ extends to a topological isomorphism

$$\varphi : B = \varprojlim B_{[a,b]} \simeq \varprojlim B_{[a^p,b^p]} = B.$$

In fact, by continuity φ is a unique topological isomorphism on B which extends φ_{inf}. Hence we establish the desired assertion. □

Definition 1.2.19. We refer to the map φ constructed in Proposition 1.2.18 as the *Frobenius automorphism* of B and define the *algebraic Fargues-Fontaine curve* to be the scheme

$$X = X_F := \text{Proj}\,(P) \qquad \text{with} \qquad P := \bigoplus_{n \geq 0} B^{\varphi = p^n}.$$

Remark. In Chapter V, we will present another incarnation of the Fargues-Fontaine curve using the theory of adic spaces developed by Huber [**Hub93, Hub94**].

PROPOSITION 1.2.20. The Fargues-Fontaine curve X is a $\mathbb{Q}_p$-scheme.

PROOF. The assertion is evident as $\mathbb{Q}_p$ naturally embeds into $B^{\varphi=1}$. □

1.3. Legendre-Newton polygons

In this subsection, we study the structures of the ring B via invariant polygons which encode the Gauss norms. For a continuous real-valued function h defined on an interval in $\mathbb{R}$, we denote its left derivative and right derivative respectively by $\partial_- h$ and $\partial_+ h$.

Definition 1.3.1. Let $\log_p$ denote the real logarithm for base p.

(1) Given an element $f \in B$, we define the *Legendre-Newton polygon* of f to be the function $\mathcal{L}_f : (0, \infty) \to \mathbb{R} \cup \{ \infty \}$ with
$$\mathcal{L}_f(s) := -\log_p \left(|f|_{p^{-s}} \right) \qquad \text{for each } s \in (0, \infty).$$

(2) Given an element $f \in B_{[a,b]}$ for some $[a, b] \subseteq (0, 1)$, we define the *Legendre-Newton polygon* of f to be the function $\mathcal{L}_f^{[a,b]} : [-\log_p(b), -\log_p(a)] \to \mathbb{R} \cup \{ \infty \}$ with
$$\mathcal{L}_f^{[a,b]}(s) := -\log_p \left(|f|_{p^{-s}} \right) \qquad \text{for each } s \in [-\log_p(b), -\log_p(a)].$$

Remark. Let us provide some motivation for studying the Legendre-Newton polygons. For a polynomial $g(z) = \sum a_n z^n$ over a field L with valuation ν_L, a useful invariant is the Newton polygon $\mathcal{N}_g$ given by the lower convex hull of the points $(n, \nu_L(a_n)) \in \mathbb{R} \times (\mathbb{R} \cup \{ \infty \})$; indeed, $\mathcal{N}_f$ contains much information about the roots of g. For an element $f \in A_{\text{inf}}[1/p, 1/[\varpi]]$ with a Teichmüller expansion $f = \sum [c_n] p^n$, we can similarly define the Newton polygon to be the largest decreasing convex function $\mathcal{N}_f : \mathbb{R} \to \mathbb{R} \cup \{ \infty \}$ with $\mathcal{N}_f(n) \leq \nu_F(c_n)$ for each $n \in \mathbb{Z}$. It turns out that $\mathcal{L}_f$ coincides with the (concave) Legendre transform of $\mathcal{N}_f$; in other words, $\mathcal{L}_f$ admits the identity
$$\mathcal{L}_f(s) = \inf_{r \in \mathbb{R}} (\mathcal{N}_f(r) + rs) \qquad \text{for each } s \in (0, \infty).$$
Hence the Legendre-Newton polygons serve as analogues of the Newton polygons for elements in B which do not necessarily admit Teichmüller expansions. In fact, as we will see in this section, the Legendre-Newton polygons turn out to be very useful for studying the elements in B via their zeros and their behavior under the Gauss norms.

LEMMA 1.3.2. For an element $f \in A_{\text{inf}}[1/p, 1/[\varpi]]$ with a Teichmüller expansion $f = \sum [c_n] p^n$, its Legendre-Newton polygon $\mathcal{L}_f$ satisfies the equality
$$\mathcal{L}_f(s) = \inf_{n \in \mathbb{Z}} (\nu_F(c_n) + ns) \qquad \text{for every } s \in (0, \infty).$$

PROOF. The assertion is evident by definition. □

Example 1.3.3. For a primitive element $\xi \in A_{\text{inf}}$ with a Teichmüller expansion $\xi = \sum [c_n] p^n$, Proposition 1.1.10 and Lemma 1.3.2 together yield the identity
$$\mathcal{L}_\xi(s) = \min(\nu_F(c_0), s) \qquad \text{for each } s \in (0, \infty).$$

Remark. In fact, an element $f \in A_{\text{inf}}$ is primitive if and only if there exists $r \in (0, \infty]$ with
$$\mathcal{L}_f(s) = \min(r, s) \qquad \text{for each } s \in (0, \infty).$$

LEMMA 1.3.4. Given elements $f, g \in B$, we have
$$\mathcal{L}_{fg}(s) = \mathcal{L}_f(s) + \mathcal{L}_g(s) \quad \text{and} \quad \mathcal{L}_{f+g}(s) \geq \min(\mathcal{L}_f(s), \mathcal{L}_g(s)) \quad \text{for each } s \in (0, \infty).$$

PROOF. The assertion is straightforward to verify by Proposition 1.2.10. □

Remark. Given a closed interval $[a, b] \subseteq (0, 1)$, we can prove a similar statement for elements of $B_{[a,b]}$ with the Legendre-Newton $[a, b]$-polygons.

PROPOSITION 1.3.5. Let f be a nonzero element in $A_{\mathrm{inf}}[1/p, 1/[\varpi]]$ with a Teichmüller expansion $f = \sum [c_n]p^n$.

(1) The function $\mathcal{L}_f$ is concave and piecewise linear with integer slopes.
(2) For each $s \in (0, \infty)$, the one-sided derivatives $\partial_- \mathcal{L}_f(s)$ and $\partial_+ \mathcal{L}_f(s)$ of $\mathcal{L}_f$ are respectively equal to the maximum and minimum elements of the set

$$T_s := \{\, n \in \mathbb{Z} : \mathcal{L}_f(s) = \nu_F(c_n) + ns \,\}.$$

PROOF. Let us fix a real number $s > 0$. Lemma 1.2.6 and Lemma 1.3.2 together imply that T_s is a nonempty finite set. Let l and r respectively denote the minimum and maximum elements of T_s. For each $n \in \mathbb{Z}$, we obtain the relation

$$\nu_F(c_l) + ls = \nu_F(c_r) + rs \leq \nu_F(c_n) + ns \tag{1.6}$$

with equality precisely when n belongs to T_s. It suffices to show that for every sufficiently small $\epsilon > 0$ we have the equalities

$$\mathcal{L}_f(s+\epsilon) = \mathcal{L}_f(s) + l\epsilon \quad \text{ and } \quad \mathcal{L}_f(s-\epsilon) = \mathcal{L}_f(s) - r\epsilon.$$

Take an integer $m < 0$ with $c_n = 0$ for each $n \leq m$ and set

$$\delta_1 := \inf_{n<l} \left(\frac{(\nu_F(c_n) + ns) - (\nu_F(c_l) + ls)}{l-n} \right) = \inf_{m<n<l} \left(\frac{(\nu_F(c_n) + ns) - (\nu_F(c_l) + ls)}{l-n} \right).$$

We see that δ_1 is positive as the inequality in the relation (1.6) is strict for each $n < l$. Consider an arbitrary real number ϵ with $0 < \epsilon < \delta_1$. For each $n < l$, we find

$$\epsilon(l-n) < \delta_1(l-n) \leq (\nu_F(c_n) + ns) - (\nu_F(c_l) + ls)$$

and in turn obtain the inequality

$$\nu_F(c_l) + l(s+\epsilon) < \nu_F(c_n) + n(s+\epsilon). \tag{1.7}$$

For each $n > l$, we use the relation (1.6) to also deduce the inequality (1.7). Therefore the Legendre-Newton polygon $\mathcal{L}_f$ satisfies the identity

$$\mathcal{L}_f(s+\epsilon) = \inf_{n\in\mathbb{Z}} (\nu_F(c_n) + n(s+\epsilon)) = \nu_F(c_l) + l(s+\epsilon) = \mathcal{L}_f(s) + l\epsilon.$$

Let us now apply Proposition 1.2.3 to choose $\lambda \in \mathbb{R}$ with $\nu_F(c_n) > \lambda$ for each $n \in \mathbb{Z}$. In addition, we set

$$u := \frac{\nu_F(c_r) - \lambda}{s/2} + r \quad \text{ and } \quad \delta_2 := \inf_{r<n<u} \left(\frac{(\nu_F(c_n) + ns) - (\nu_F(c_r) + rs)}{n-r} \right).$$

We see that δ_2 is positive as the inequality in the relation (1.6) is strict for each $n > r$. Consider an arbitrary real number ϵ with $0 < \epsilon < \min(s/2, \delta_2)$. For each $n > u$ we have

$$\nu_F(c_r) - \nu_F(c_n) < \nu_F(c_r) - \lambda = (u-r)s/2 < (n-r)(s-\epsilon)$$

and thus establish the inequality

$$\nu_F(c_r) + r(s-\epsilon) < \nu_F(c_n) + n(s-\epsilon). \tag{1.8}$$

For each $n \in \mathbb{Z}$ with $r < n < u$, we find

$$\epsilon(n-r) < \delta_2(n-r) \leq (\nu_F(c_n) + ns) - (\nu_F(c_r) + rs)$$

and in turn obtain the inequality (1.8). For each $n < r$, we use the relation (1.6) to also deduce the inequality (1.8). Therefore the Legendre-Newton polygon $\mathcal{L}_f$ satisfies the identity

$$\mathcal{L}_f(s-\epsilon) = \inf_{n\in\mathbb{Z}} (\nu_F(c_n) + n(s-\epsilon)) = \nu_F(c_r) + r(s-\epsilon) = \mathcal{L}_f(s) - r\epsilon.$$

The desired assertion is now evident. □

PROPOSITION 1.3.6. Let f be a nonzero element of $B_{[a,b]}$ for some $[a,b] \subseteq (0,1)$.

(1) For a sequence (f_n) in $A_{\inf}[1/p, 1/[\varpi]]$ which converges to f under the Gauss a-norm and the Gauss b-norm, there exists an integer $m > 0$ with

$$\mathcal{L}_{f_n}^{[a,b]} = \mathcal{L}_f^{[a,b]} \quad \text{for each } n > m.$$

(2) The function $\mathcal{L}_f^{[a,b]}$ is concave and piecewise linear with integer slopes.

PROOF. We write $l := -\log_p(b)$ and $r := -\log_p(a)$. For a sequence (f_n) in $A_{\inf}[1/p, 1/[\varpi]]$ which converges to f under the Gauss a-norm and the Gauss b-norm, Lemma 1.2.12 yields the equality

$$\mathcal{L}_f^{[a,b]}(s) = \lim_{n\to\infty} \mathcal{L}_{f_n}(s) \quad \text{for each } s \in [l,r]. \tag{1.9}$$

In addition, either $\mathcal{L}_f^{[a,b]}(l)$ or $\mathcal{L}_f^{[a,b]}(r)$ is finite as f is nonzero. Let us only consider the case where $\mathcal{L}_f^{[a,b]}(l)$ is finite, as the same argument works for the case where $\mathcal{L}_f^{[a,b]}(r)$ is finite. Take an integer $u > 0$ with

$$\mathcal{L}_{f_n - f_u}(l) > \mathcal{L}_f^{[a,b]}(l) + 1 > \mathcal{L}_{f_n}(l) \quad \text{for each } n \geq u.$$

For every integer $n \geq u$, we apply Proposition 1.3.5 to choose a real number $\delta_n > 0$ with

$$\mathcal{L}_{f_n - f_u}(l+\epsilon) > \mathcal{L}_f^{[a,b]}(l) + 1 > \mathcal{L}_{f_n}(l+\epsilon) \quad \text{for each } \epsilon \in (-\delta_n, \delta_n)$$

and in turn use Lemma 1.3.4 to find

$$\mathcal{L}_{f_n}(l+\epsilon) = \mathcal{L}_{f_u}(l+\epsilon) \quad \text{for each } \epsilon \in (-\delta_n, \delta_n).$$

Hence we obtain the equalities

$$\mathcal{L}_{f_n}(l) = \mathcal{L}_{f_u}(l) \quad \text{and} \quad \partial_+\mathcal{L}_{f_n}(l) = \partial_+\mathcal{L}_{f_u}(l) \quad \text{for every } n > u.$$

Let us now set

$$\omega := \max(\mathcal{L}_{f_u}(l), \mathcal{L}_{f_u}(l) + \partial_+\mathcal{L}_{f_u}(l)(r-l)).$$

Proposition 1.3.5 implies that each $\mathcal{L}_{f_n}$ with $n > u$ satisfies the inequality

$$\mathcal{L}_{f_n}(s) \leq \omega \quad \text{for every } s \in [l,r].$$

Moreover, if we take an integer $m > u$ with

$$\mathcal{L}_{f_n - f_m}(l) > \omega \quad \text{and} \quad \mathcal{L}_{f_n - f_m}(r) > \omega \quad \text{for each } n > m,$$

we see by Lemma 1.2.12 that each $\mathcal{L}_{f_n - f_m}$ with $n > m$ satisfies the inequality

$$\mathcal{L}_{f_n - f_m}(s) > \omega \quad \text{for every } s \in [l,r]$$

and in turn deduce from Lemma 1.3.4 that each $\mathcal{L}_{f_n}$ with $n > u$ admits the identity

$$\mathcal{L}_{f_n}(s) = \mathcal{L}_{f_m}(s) \quad \text{for every } s \in [l,r].$$

Hence we establish statement (1) by the equality (1.9). Statement (2) immediately follows from statement (1) and Proposition 1.3.5. □

PROPOSITION 1.3.7. For every nonzero $f \in B$, the function $\mathcal{L}_f$ is concave and piecewise linear with integer slopes.

PROOF. It suffices to show that $\mathcal{L}_f$ is concave and piecewise linear with integer slopes on every closed interval $[a,b] \subseteq (0,1)$. If we write $f_{a,b}$ for the image of f under the natural homomorphism $B \to B_{[a,b]}$, we identify $\mathcal{L}_{f_{a,b}}^{[a,b]}$ with the restriction of $\mathcal{L}_f$ on $[-\log_p(b), -\log_p(a)]$. Therefore we deduce the desired assertion from Proposition 1.3.6. □

PROPOSITION 1.3.8. The natural map $\mathcal{O}_F \to B$ sending each $c \in \mathcal{O}_F$ to $[c] \in B$ is continuous.

PROOF. Let us choose an untilt C of F in characteristic 0. The natural map $\mathcal{O}_F \to B$ composed with $\widehat{\theta_C}$ coincides with the sharp map associated to C on $\mathcal{O}_F$, which is continuous by Proposition 2.1.8 in Chapter III. Hence we deduce the desired assertion from the fact that $\widehat{\theta_C}$ is continuous and open by construction. □

Remark. We can extend Proposition 1.3.8 to get a continuous map $\prod_{n=0}^{\infty} \mathcal{O}_F \to B$ which sends each $(c_n) \in \prod_{n=0}^{\infty} \mathcal{O}_F$ to $\sum_{n=0}^{\infty} [c_n]p^n \in B$.

PROPOSITION 1.3.9. Given an element $f \in B$ and an integer $n \geq 0$ with $|f|_\rho \leq \rho^n$ for every $\rho \in (0,1)$, there exists an element $c \in \mathcal{O}_F$ with $|f - [c]p^n|_\rho \leq \rho^{n+1}$ for every $\rho \in (0,1)$.

PROOF. We may replace f by f/p^n to assume the equality $n = 0$. Since the assertion is evident for $f = 0$, we may also assume that f is nonzero. Choose a sequence $(\widetilde{f_m})$ in $A_{\text{inf}}[1/p, 1/[\varpi]]$ which converges to f under all Gauss norms. For each $m \geq 1$, we write

$$\widetilde{f_m} = f_m + \sum_{i<0} [c_{m,i}]p^i \quad \text{with } c_{m,i} \in F \text{ and } f_m \in A_{\text{inf}}[1/[\varpi]].$$

We see that the sequence (f_m) converges to f under all Gauss norms; indeed, if we consider arbitrary real numbers $\rho, \epsilon \in (0,1)$, for each $m \gg 0$ we obtain the equality $\left|\widetilde{f_m}\right|_{\epsilon\rho} = |f|_{\epsilon\rho}$ by Proposition 1.3.6 and in turn find

$$\left|\widetilde{f_m} - f_m\right|_\rho = \sup_{i<0} \left(|c_{m,i}|\, \rho^i\right) \leq \sup_{i<0} \left(\epsilon^{-i}\right) \sup_{i<0} \left(|c_{m,i}|\, \epsilon^i \rho^i\right) \leq \epsilon \left|\widetilde{f_m}\right|_{\epsilon\rho} = \epsilon\, |f|_{\epsilon\rho} \leq \epsilon.$$

For each $m \geq 1$, let us denote by c_m the image of f_m under the natural map $W(F) \twoheadrightarrow F$. Each $f_{m+1} - f_m$ has the first term in the Teichmüller expansion given by $[c_{m+1} - c_m]$ and thus satisfies the inequality

$$|c_{m+1} - c_m| \leq |f_{m+1} - f_m|_\rho \quad \text{for every } \rho \in (0,1).$$

We see that the sequence (c_m) converges to an element $c \in F$ for being Cauchy. Moreover, since each f_m has the first term in the Teichmüller expansion given by $[c_m]$, for every $\rho \in (0,1)$ we apply Proposition 1.3.6 to find

$$|c_m| \leq |f_m|_\rho = |f|_\rho \leq 1 \quad \text{for every } m \gg 0.$$

Hence we deduce that c lies in $\mathcal{O}_F$.

Let us now set $g_m := f_m - [c_m] \in A_{\text{inf}}[1/[\varpi]]$ for each $m \geq 1$ and take $g := f - [c] \in B$. We wish to establish the inequality $|g|_\rho \leq \rho$ for every $\rho \in (0,1)$. If g is zero, the inequality evidently holds. We henceforth assume that g is nonzero. Proposition 1.3.8 implies that the sequence (g_m) converges to g under all Gauss norms. Hence we may remove finitely many terms from the sequence (g_m) to assume that each g_m is nonzero. Since the Teichmüller expansion of each g_m only involves positive powers of p, each $\mathcal{L}_{g_m}$ is piecewise linear with positive integer slopes by Proposition 1.3.5. Now Proposition 1.3.5 and Proposition 1.3.7 together show that $\mathcal{L}_g$ is piecewise linear with positive integer slopes. Meanwhile, we apply Lemma 1.3.4 to find

$$\mathcal{L}_g(s) \geq \min\left(\mathcal{L}_f(s), \mathcal{L}_{[c]}(s)\right) = \min\left(-\log_p\left(|f|_{p^{-s}}\right), -\log_p(|c|)\right) \geq 0 \quad \text{for each } s \in (0,\infty).$$

Hence we have $\mathcal{L}_g(s) \geq s$ for every $s \in (0,\infty)$, or equivalently $|g|_\rho \leq \rho$ for every $\rho \in (0,1)$. □

PROPOSITION 1.3.10. Let f be a nonzero element in B.

(1) The element f lies in A_{inf} if and only if we have $|f|_\rho \le 1$ for every $\rho \in (0,1)$.
(2) The element f lies in $A_{\text{inf}}[1/p]$ if and only if there exists an integer n with $|f|_\rho \le \rho^n$ for every $\rho \in (0,1)$.
(3) The element f lies in $A_{\text{inf}}[1/[\varpi]]$ if and only if there exists a real number $\lambda > 0$ with $|f|_\rho \le \lambda$ for every $\rho \in (0,1)$.
(4) The element f lies in $A_{\text{inf}}[1/p, 1/[\varpi]]$ if and only if there exist a real number $\lambda > 0$ and an integer n with $|f|_\rho \le \lambda\rho^n$ for every $\rho \in (0,1)$.

PROOF. If f lies in A_{inf}, it evidently satisfies the inequality $|f|_\rho \le 1$ for every $\rho \in (0,1)$. Conversely, if we have $|f|_\rho \le 1$ for every $\rho \in (0,1)$, we apply Proposition 1.3.9 to inductively construct a sequence (c_i) in $\mathcal{O}_F$ with

$$\left| f - \sum_{i=0}^{n-1} [c_i] p^i \right|_\rho \le \rho^n \qquad \text{for every } n \ge 0 \text{ and } \rho \in (0,1)$$

and in turn find $f \in A_{\text{inf}}$. Therefore we establish statement (1). In addition, f lies in $A_{\text{inf}}[1/p]$ if and only if there exists an integer n with $p^n f \in A_{\text{inf}}$, or equivalently $|f|_\rho \le |p|_\rho^{-n} = \rho^{-n}$ for every $\rho \in (0,1)$ as asserted in statement (2). Similarly, f lies in $A_{\text{inf}}[1/[\varpi]]$ if and only if there exists an integer n with $[\varpi^n] f \in A_{\text{inf}}$, or equivalently $|f|_\rho \le |[\varpi]|_\rho^{-n} = |\varpi|^{-n}$ for every $\rho \in (0,1)$ as asserted in statement (3). Likewise, f lies in $A_{\text{inf}}[1/p, 1/[\varpi]]$ if and only if there exist integers m and n with $p^n[\varpi]^m f \in A_{\text{inf}}$, or equivalently $|f|_\rho \le |[\varpi]^{-m} p^{-n}|_\rho = |\varpi|^{-m} \rho^{-n}$ for every $\rho \in (0,1)$ as asserted in statement (4). $\square$

LEMMA 1.3.11. Every $f \in B$ satisfies the equalities

$$|\varphi(f)|_{\rho^p} = |f|_\rho^p \qquad \text{and} \qquad |pf|_\rho = \rho\, |f|_\rho \qquad \text{for each } \rho \in (0,1).$$

PROOF. If f lies in $A_{\text{inf}}[1/p, 1/[\varpi]]$, the assertion is evident by construction. For the general case, we obtain the assertion by continuity. $\square$

PROPOSITION 1.3.12. The $\mathbb{Q}_p$-vector space $B^{\varphi = p^n}$ is trivial for every $n < 0$.

PROOF. Take an arbitrary element $f \in B^{\varphi = p^n}$. Lemma 1.3.11 yields the equality

$$p\mathcal{L}_f(s) = \mathcal{L}_{\varphi(f)}(ps) = \mathcal{L}_{p^n f}(ps) = nps + \mathcal{L}_f(ps) \qquad \text{for each } s \in (0,\infty).$$

If f is nonzero, we apply Proposition 1.3.7 to obtain the relation

$$p\partial_+\mathcal{L}_f(s) = np + p\partial_+\mathcal{L}_f(ps) \le np + p\partial_+\mathcal{L}_f(s) \qquad \text{for each } s \in (0,\infty),$$

which is impossible as n is negative. Hence we deduce that $B^{\varphi = p^n}$ is trivial as desired. $\square$

PROPOSITION 1.3.13. The ring $B^{\varphi = 1}$ is canonically isomorphic to $\mathbb{Q}_p$.

PROOF. The field $\mathbb{Q}_p$ embeds into $B^{\varphi=1}$ via an identification $\mathbb{Q}_p \cong A_{\text{inf}}[1/p, 1/[\varpi]]^{\varphi=1}$ as easily seen by Teichmüller expansions. Hence it suffices to show that every nonzero $f \in B^{\varphi=1}$ lies in $\mathbb{Q}_p$. We apply Lemma 1.3.11 to find

$$p\mathcal{L}_f(s) = \mathcal{L}_{\varphi(f)}(ps) = \mathcal{L}_f(ps) \qquad \text{for each } s \in (0,\infty)$$

and in turn obtain the equality $p\partial_+\mathcal{L}_f(s) = p\partial_+\mathcal{L}_f(ps)$ for each $s \in (0,\infty)$. Now we see by Proposition 1.3.7 that $\mathcal{L}_f$ is linear with an integer slope, which means that there exist some $n \in \mathbb{Z}$ and $r \in \mathbb{R}$ with $\mathcal{L}_f(s) = ns + r$ for each $s \in (0,\infty)$, or equivalently $|f|_\rho = p^{-r}\rho^n$ for each $\rho \in (0,1)$. Hence Proposition 1.3.10 implies that f lies in $A_{\text{inf}}[1/p, 1/[\varpi]]^{\varphi=1} \cong \mathbb{Q}_p$. $\square$

PROPOSITION 1.3.14. Let $[a, b]$ be a closed subinterval of $(0, 1)$.

(1) The ring $B_{[a,b]}$ is an integral domain.
(2) The natural ring homomorphism $B \to B_{[a,b]}$ is injective.

PROOF. Consider arbitrary nonzero elements $f, g \in B_{[a,b]}$. Proposition 1.3.6 implies that both $\mathcal{L}_f^{[a,b]}$ and $\mathcal{L}_g^{[a,b]}$ take finite values, which means that both $|f|_\rho$ and $|g|_\rho$ are nonzero for each $\rho \in [a, b]$. Hence we deduce from Proposition 1.2.10 that fg is nonzero and in turn establish statement (1).

It remains to prove statement (2). Take an arbitrary nonzero element $h \in B$ and denote by $h_{a,b}$ its image under the natural map $B \to B_{[a,b]}$. We may identify $\mathcal{L}_{h_{a,b}}^{[a,b]}$ with the restriction of $\mathcal{L}_h$ on $[-\log_p(b), -\log_p(a)]$. Since $\mathcal{L}_h$ takes finite values by Proposition 1.3.7, we see that $h_{a,b}$ is nonzero and thus obtain statement (2). □

Remark. For every closed interval $[a', b']$ with $[a, b] \subseteq [a', b'] \subseteq (0, 1)$, we can similarly show that the natural ring homomorphism $B_{[a',b']} \to B_{[a,b]}$ is injective.

LEMMA 1.3.15. Let f and g be elements in B. If f is divisible by g in $B_{[a,b]}$ for every closed interval $[a, b] \subseteq (0, 1)$, then f is divisible by g in B.

PROOF. For every closed interval $[a, b] \subseteq (0, 1)$, we deduce from Proposition 1.3.14 that there exists a unique element $h_{a,b} \in B_{[a,b]}$ with $f = gh_{a,b}$. Hence we obtain an element $h \in B$ with $f = gh$ as desired. □

PROPOSITION 1.3.16. Let y be an element in Y and C be a representative of y. Every $f \in B$ with $f(y) = 0$ is divisible by every primitive element $\xi \in \ker(\theta_C)$.

PROOF. Take an arbitrary closed interval $[a, b] \subseteq (0, 1)$. By Lemma 1.3.15, it suffices to prove that f is divisible by ξ in $B_{[a,b]}$. Choose a sequence (f_n) in $A_{\text{inf}}[1/p, 1/[\varpi]]$ which converges to f with respect to the Gauss a-norm and the Gauss b-norm. Proposition 1.1.6 shows that each f_n admits an expression $f_n(y) = c_n^\sharp$ for some $c_n \in F$. Since we have

$$\lim_{n\to\infty} |c_n| = \lim_{n\to\infty} \left|c_n^\sharp\right|_C = \lim_{n\to\infty} |f_n(y)|_C = |f(y)|_C = 0,$$

the sequence $([c_n])$ converges to 0 with respect to the Gauss a-norm and the Gauss b-norm. Hence we may replace each f_n by $f_n - [c_n]$ to assume the equality $f_n(y) = 0$.

Proposition 1.1.25 yields an element $g_n \in A_{\text{inf}}[1/p, 1/[\varpi]]$ with $f_n = \xi g_n$ for each $n \geq 1$. Moreover, for every $\rho \in [a, b]$ we apply Proposition 1.2.10 to obtain the relation

$$\lim_{n\to\infty} |g_{n+1} - g_n|_\rho = \frac{1}{|\xi|_\rho} \cdot \lim_{n\to\infty} |\xi(g_{n+1} - g_n)|_\rho = \frac{1}{|\xi|_\rho} \cdot \lim_{n\to\infty} |f_{n+1} - f_n|_\rho = 0,$$

which means that the sequence (g_n) is Cauchy with respect to the Gauss ρ-norm. Now the sequence (g_n) gives rise to an element $g \in B_{[a,b]}$ with $f = \xi g$ as desired. □

PROPOSITION 1.3.17. Given an untilt C of F in characteristic 0, every primitive $\xi \in \ker(\theta_C)$ generates $\ker(\widehat{\theta_C})$.

PROOF. The assertion immediately follows from Proposition 1.3.16. □

Remark. Theorem 1.1.27 and Proposition 1.3.17 together show that Y admits a natural embedding into the set of closed maximal ideals in B. It turns out that this embedding is a bijection.

PROPOSITION 1.3.18. Given an untilt C of F in characteristic 0, we have
$$A_{\inf}[1/p] \cap \ker(\widehat{\theta_C})^i = \ker(\theta_C[1/p])^i \quad \text{for each } i \geq 1.$$

PROOF. The assertion for $i = 1$ is evident by the fact that $\widehat{\theta_C}$ restricts to $\theta_C[1/p]$. Let us now proceed by induction on i. Since we have
$$A_{\inf}[1/p] \cap \ker(\widehat{\theta_C})^i \supseteq \ker(\theta_C[1/p])^i,$$
we only need to prove that every $a \in A_{\inf}[1/p] \cap \ker(\widehat{\theta_C})^i$ belongs to $\ker(\theta_C[1/p])^i$. By Proposition 1.1.25 and Proposition 1.3.17, there exists a primitive element $\xi \in A_{\inf}$ which generates $\ker(\widehat{\theta_C})$ and $\ker(\theta_C[1/p])$. We write $a = \xi^i b$ for some $b \in B$ and use the relation
$$A_{\inf}[1/p] \cap \ker(\widehat{\theta_C})^i \subseteq A_{\inf}[1/p] \cap \ker(\widehat{\theta_C})^{i-1} = \ker(\theta_C[1/p])^{i-1}$$
given by the induction hypothesis to find $c \in A_{\inf}[1/p]$ with $a = \xi^{i-1}c$. Now we have
$$0 = a - a = \xi^i b - \xi^{i-1} c = \xi^{i-1}(\xi b - c)$$
and thus apply Proposition 1.3.14 to obtain the relation
$$c = \xi b \in A_{\inf}[1/p] \cap \ker(\widehat{\theta_C}) = \ker(\theta_C[1/p]),$$
which in particular implies that $a = \xi^{i-1}c$ lies in $\ker(\theta_C[1/p])^i$ as desired. □

Definition 1.3.19. Given an element $y \in Y$ represented by an untilt C of F, we define the *de Rham local ring at* y to be
$$B_{\mathrm{dR}}^+(y) := \varprojlim_i A_{\inf}[1/p]/\ker(\theta_C[1/p])^i.$$

Remark. Theorem 1.1.27 shows that $B_{\mathrm{dR}}^+(y)$ does not depend on the representative C.

PROPOSITION 1.3.20. Let y be an element in Y and C be a representative of y.

(1) The ring $B_{\mathrm{dR}}^+(y)$ is a complete discrete valuation ring with residue field C.
(2) Every primitive element in $\ker(\theta_C)$ is a uniformizer of $B_{\mathrm{dR}}^+(y)$.
(3) There exists a natural isomorphism
$$B_{\mathrm{dR}}^+(y) \cong \varprojlim_i B/\ker(\widehat{\theta_C})^i.$$

PROOF. Given a p-adic field K, all results from the first part of §2.2 in Chapter III rely only on the fact that $\mathbb{C}_K$ is an algebraically closed perfectoid field. Since C is algebraically closed as noted in Proposition 1.1.6, these results remain valid with C in place of $\mathbb{C}_K$. Hence we establish statement (1) by Proposition 2.2.17 in Chapter III and deduce statement (2) from Proposition 1.1.25.

It remains to verify statement (3). By Proposition 1.1.25 and Proposition 1.3.17, there exists a primitive element $\xi \in A_{\inf}$ which generates $\ker(\widehat{\theta_C})$ and $\ker(\theta_C[1/p])$. Hence we obtain a natural map
$$B_{\mathrm{dR}}^+(y) = \varprojlim_i A_{\inf}[1/p]/\xi^i A_{\inf}[1/p] \longrightarrow \varprojlim_i B/\xi^i B = \varprojlim_i B/\ker(\widehat{\theta_C})^i. \tag{1.10}$$
Proposition 1.3.18 shows that the map (1.10) is injective. Moreover, since we have
$$A_{\inf}[1/p]/\xi A_{\inf}[1/p] \cong C \cong B/\xi B,$$
the map (1.10) is surjective by a general fact stated in the Stacks project [**Sta**, Tag 0315]. Now we deduce that the natural map (1.10) is an isomorphism, thereby completing the proof. □

Definition 1.3.21. Given a nonzero element $f \in B$, we define the *vanishing order* of f at an element $y \in Y$ to be the valuation of f in $B_{\mathrm{dR}}^{+}(y)$, denoted by $\mathrm{ord}_y(f)$.

LEMMA 1.3.22. Given nonzero elements $f, g \in B$, we have

$$\mathrm{ord}_y(fg) = \mathrm{ord}_y(f) + \mathrm{ord}_y(g) \quad \text{for each } y \in Y.$$

PROOF. The assertion is evident by definition. □

PROPOSITION 1.3.23. Let y be an element in Y and C be a representative of y. For every nonzero $f \in B$, we have $f(y) = 0$ if and only if $\mathrm{ord}_y(f)$ is positive.

PROOF. The assertion immediately follows from Proposition 1.3.20. □

Remark. By Proposition 1.3.23, we can make sense of whether f vanishes at y without choosing a representative.

PROPOSITION 1.3.24. Let f be a nonzero element in B and $[a, b]$ be a closed interval in $(0, 1)$.

(1) The vanishing order of f at every $y \in Y_{[a,b]}$ is finite.
(2) The set $Z_{[a,b]} := \left\{ y \in Y_{[a,b]} : \mathrm{ord}_y(f) \neq 0 \right\}$ is finite.

PROOF. Let us write $l := -\log_p(b)$ and $r := -\log_p(a)$. Proposition 1.3.7 implies that $n := \partial_- \mathcal{L}_f(l) - \partial_+ \mathcal{L}_f(r)$ is a nonnegative integer. It suffices to prove the inequality

$$\sum_{y \in Z_{[a,b]}} \mathrm{ord}_y(f) \leq n. \tag{1.11}$$

Suppose for contradiction that this inequality does not hold. We apply Proposition 1.3.16, Proposition 1.3.20, and Lemma 1.3.22 to write

$$f = \xi_1 \xi_2 \cdots \xi_{n+1} g$$

for some $g \in B$ and primitive elements $\xi_1, \cdots, \xi_{n+1} \in A_{\mathrm{inf}}$. Since each ξ_i vanishes at a unique element in $Y_{[a,b]}$, Example 1.2.2 and Example 1.3.3 together yield the identity

$$\partial_- \mathcal{L}_{\xi_i}(l) - \partial_+ \mathcal{L}_{\xi_i}(r) = 1 - 0 = 1.$$

In addition, by Proposition 1.3.7 we have

$$\partial_- \mathcal{L}_g(l) - \partial_+ \mathcal{L}_g(r) \geq 0.$$

Now we use Lemma 1.3.4 to find

$$n = \partial_- \mathcal{L}_f(l) - \partial_+ \mathcal{L}_f(r) = \sum_{i=1}^{n+1} \left(\partial_- \mathcal{L}_{\xi_i}(l) - \partial_+ \mathcal{L}_{\xi_i}(r) \right) + \left(\partial_- \mathcal{L}_g(l) - \partial_+ \mathcal{L}_g(r) \right) \geq n + 1,$$

thereby obtaining a contradiction as desired. □

Remark. It turns out that the inequality (1.11) is an equality.

PROPOSITION 1.3.25. The ring B is naturally a subring of $B_{\mathrm{dR}}^{+}(y)$ for every $y \in Y$.

PROOF. Since $\mathrm{ord}_y(f)$ is finite for each nonzero $f \in B$ as noted in Proposition 1.3.24, the assertion follows from Proposition 1.3.20. □

Definition 1.3.26. Given a nonzero element $f \in B$, its *Weil divisor* on Y is the formal sum

$$\mathrm{Div}_Y(f) := \sum_{y \in Y} \mathrm{ord}_y(f) \cdot y.$$

Remark. We may regard $\mathrm{Div}_Y(f)$ as a locally finite sum by Proposition 1.3.24.

1.4. The logarithm and closed points

In this subsection, we study the graded ring $P = \bigoplus B^{\varphi=p^n}$ to establish some fundamental properties of the Fargues-Fontaine curve. Throughout this section, we write $\mathfrak{m}_F^* := \mathfrak{m}_F \setminus \{ 0 \}$.

PROPOSITION 1.4.1. There exists a group homomorphism $\log : 1 + \mathfrak{m}_F \to B^{\varphi=p}$ with

$$\log(\varepsilon) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon] - 1)^n}{n} \quad \text{for every } \varepsilon \in 1 + \mathfrak{m}_F. \tag{1.12}$$

PROOF. Given $\varepsilon \in 1 + \mathfrak{m}_F$ and $\rho \in (0, 1)$, we write $[\varepsilon] - 1 = \sum [c_n] p^n$ with $c_n \in \mathcal{O}_F$ to find

$$\left|[\varepsilon] - 1\right|_\rho \leq \max(|c_0|, \rho) = \max(|\varepsilon - 1|, \rho) < 1.$$

Hence we obtain a map $\log : 1 + \mathfrak{m}_F \to B$ with the identity (1.12). Moreover, since we have $\log(uv) = \log(u) + \log(v)$ as formal power series, we deduce from the multiplicativity of Teichmüller lifts that log is a group homomorphism. Now for every $\varepsilon \in 1 + \mathfrak{m}_F$ we find

$$\varphi(\log(\varepsilon)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\varphi([\varepsilon]) - 1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([\varepsilon^p] - 1)^n}{n} = \log(\varepsilon^p) = p \log(\varepsilon),$$

thereby completing the proof. □

Remark. We will see in Proposition 1.4.17 that log is a $\mathbb{Q}_p$-linear isomorphism.

Definition 1.4.2. We refer to the map $\log : 1 + \mathfrak{m}_F \to B^{\varphi=p}$ given by Proposition 1.4.1 as the *tilted logarithm.*

PROPOSITION 1.4.3. Let C be an untilt of F and denote by $\mathfrak{m}_C$ the maximal ideal of $\mathcal{O}_C$.

(1) An element $c \in \mathcal{O}_F$ lies in $1 + \mathfrak{m}_F$ if and only if $c^\sharp$ lies in $1 + \mathfrak{m}_C$.
(2) If C is in characteristic 0, there exists a commutative diagram

$$\begin{array}{ccc} 1 + \mathfrak{m}_F & \xrightarrow{\ \log\ } & B^{\varphi=p} \\ {\scriptstyle \varepsilon \mapsto \varepsilon^\sharp} \big\downarrow & & \big\downarrow {\scriptstyle \widehat{\theta_C}} \\ 1 + \mathfrak{m}_C & \xrightarrow{\ \log_{\mu_{p^\infty}}\ } & C \end{array}$$

where all maps are group homomorphisms.

PROOF. Take an arbitrary element $c \in \mathcal{O}_F$. Proposition 2.1.10 in Chapter III yields an element $a \in \mathcal{O}_C$ with $c^\sharp - 1 = (c - 1)^\sharp + pa$. If c belongs to $1 + \mathfrak{m}_F$, we have

$$\nu_C(c^\sharp - 1) \geq \min\left(\nu_C\left((c - 1)^\sharp\right), \nu_C(pa)\right) = \min(\nu_F(c - 1), \nu_C(pa)) > 0$$

and consequently see that $c^\sharp$ lies in $1 + \mathfrak{m}_C$. Conversely, if $c^\sharp$ belongs to $1 + \mathfrak{m}_C$, we find

$$\nu_F(c - 1) = \nu_C((c - 1)^\sharp) \geq \min\left(\nu_C(c^\sharp - 1), \nu_C(pa)\right) > 0$$

and in turn see that c lies in $1 + \mathfrak{m}_F$. Hence we establish statement (1).

Now statement (1) and Proposition 1.1.6 together show that $1 + \mathfrak{m}_F$ maps onto $1 + \mathfrak{m}_C$ under the sharp map. If C is in characteristic 0, every $\varepsilon \in 1 + \mathfrak{m}_F$ yields the identity

$$\widehat{\theta_C}(\log(\varepsilon)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\widehat{\theta_C}([\varepsilon]) - 1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(\varepsilon^\sharp - 1)^n}{n} = \log_{\mu_{p^\infty}}(\varepsilon^\sharp)$$

where the last equality follows from Example 3.2.18 in Chapter II. Moreover, since C is algebraically closed by Proposition 1.1.6, the map $\log_{\mu_{p^\infty}}$ is a surjective homomorphism by Proposition 3.2.20 in Chapter II. Therefore we obtain statement (2). □

Definition 1.4.4. For every $\varepsilon \in 1 + \mathfrak{m}_F^*$, its associated *cyclotomic element* in A_{inf} is

$$\xi_\varepsilon := \frac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1} = 1 + [\varepsilon^{1/p}] + \cdots + [\varepsilon^{(p-1)/p}].$$

PROPOSITION 1.4.5. Let ε be an element in $1 + \mathfrak{m}_F^*$.

(1) The element $\xi_\varepsilon \in A_{\text{inf}}$ is strongly primitive.
(2) The element $\xi_\varepsilon \in A_{\text{inf}}$ divides $[\varepsilon] - 1$ but not $[\varepsilon^{1/p}] - 1$.
(3) The element $y_\varepsilon \in Y$ represented by C_{ξ_ε} admits the equality $\text{ord}_{y_\varepsilon}(\log(\varepsilon)) = 1$.

PROOF. Let us write $k := \mathcal{O}_F/\mathfrak{m}_F$ for the residue field of F. In addition, for every $c \in \mathcal{O}_F$ we denote by $\overline{c}$ its image under the natural map $\mathcal{O}_F \twoheadrightarrow k$. Theorem 2.3.1 in Chapter II yields a ring homomorphism $\eta : A_{\text{inf}} \to W(k)$ with

$$\eta\left(\sum [c_n]p^n\right) = \sum [\overline{c_n}]p^n \quad \text{for each } c_n \in \mathcal{O}_F.$$

We find $\eta(\xi_\varepsilon) = p$ by the identity $\overline{\varepsilon^{1/p}} = \overline{\varepsilon}^{1/p} = 1$ and thus obtain a Teichmüller expansion

$$\xi_\varepsilon = [m_0] + [m_1 + 1]p + \sum_{n \geq 2} [m_n]p^n \quad \text{with } m_n \in \mathfrak{m}_F.$$

Since we have

$$m_0 = 1 + \varepsilon^{1/p} + \cdots + \varepsilon^{(p-1)/p} = \frac{\varepsilon - 1}{\varepsilon^{1/p} - 1} \neq 0,$$

we deduce statement (1) from Proposition 1.1.10.

It is evident by construction that ξ_ε divides $[\varepsilon] - 1$. If ξ_ε divides $[\varepsilon^{1/p}] - 1$, we see that $\xi_\varepsilon = 1 + [\varepsilon^{1/p}] + \cdots + [\varepsilon^{(p-1)/p}]$ should divide p, which contradicts Proposition 1.1.11. Hence we deduce that ξ_ε does not divide $[\varepsilon^{1/p}] - 1$ and in turn obtain statement (2).

Now Proposition 1.3.20 shows that $[\varepsilon] - 1 = \xi_\varepsilon([\varepsilon^{1/p}] - 1)$ is a uniformizer of $B_{\text{dR}}^+(y_\varepsilon)$. In addition, $\log(\varepsilon)$ is divisible by $[\varepsilon] - 1$ but not by $([\varepsilon] - 1)^2$. Therefore we find $\text{ord}_{y_\varepsilon}(\log(\varepsilon)) = 1$ as asserted in statement (3). □

Remark. The main insight behind Proposition 1.4.5 is that ε should give rise to a system of p-power primitive roots of unity in some untilt of F, as foreshadowed by our discussion in Chapter III.

Definition 1.4.6. Given an element $\varepsilon \in 1 + \mathfrak{m}_F^*$, the *untilt class of F* associated to ε is the element $y_\varepsilon \in Y$ represented by C_{ξ_ε}.

PROPOSITION 1.4.7. There exists a bijection $(1 + \mathfrak{m}_F^*)/\mathbb{Z}_p^\times \xrightarrow{\sim} Y$ which maps the $\mathbb{Z}_p^\times$-orbit of an element $\varepsilon \in 1 + \mathfrak{m}_F^*$ to $y_\varepsilon \in Y$.

PROOF. Let y be an arbitrary element in Y and C be a representative of y. Since C is algebraically closed as noted in Proposition 1.1.6, it admits a system of primitive p-power roots of unity which is unique up to $\mathbb{Z}_p^\times$-multiple. This system yields a unique $\mathbb{Z}_p^\times$-orbit of an element $\varepsilon \in \mathcal{O}_F$ with $\varepsilon^\sharp = 1$ and $(\varepsilon^{1/p})^\sharp \neq 1$ via the tilting isomorphism $F \simeq C^\flat$. We note that ε lies in $1 + \mathfrak{m}_F^*$ by Proposition 1.4.3 and in turn find $y = y_\varepsilon$ by the equality

$$\theta_C(\xi_\varepsilon) = \frac{\theta_C([\varepsilon] - 1)}{\theta_C([\varepsilon^{1/p}] - 1)} = \frac{\varepsilon^\sharp - 1}{(\varepsilon^{1/p})^\sharp - 1} = 0$$

In addition, every $\zeta \in 1 + \mathfrak{m}_F^*$ with $y = y_\zeta$ should be a $\mathbb{Z}_p^\times$-multiple of ε as it satisfies the relations $\zeta^\sharp = 1$ and $(\zeta^{1/p})^\sharp \neq 1$ by Proposition 1.4.5. Now the desired assertion is evident. □

Definition 1.4.8. Let φ_F denote the Frobenius automorphism of F.

(1) Given an untilt C of F, its *Frobenius twist* $\phi(C)$ is the perfectoid field C with the topological isomorphism $\iota_{\phi(C)} = \iota_C \circ \varphi_F$.
(2) The *Frobenius action* on Y is the bijection $\phi : Y \to Y$ induced by Frobenius twists.

LEMMA 1.4.9. For every untilt C of F in characteristic 0, we have $\widehat{\theta_{\phi(C)}} = \widehat{\theta_C} \circ \varphi$

PROOF. We observe the identity $\widetilde{\theta_{\phi(C)}} = \widetilde{\theta_C} \circ \varphi$ by construction and in turn obtain the desired assertion by continuity. □

PROPOSITION 1.4.10. Every $\varepsilon \in 1 + \mathfrak{m}_F^*$ yields the identity $\phi^{-1}(y_\varepsilon) = y_{\varepsilon^p}$.

PROOF. Let us write $C_\varepsilon := C_{\xi_\varepsilon}$ for notational symplicity. We find

$$\theta_{\phi^{-1}(C_\varepsilon)}(\xi_{\varepsilon^p}) = \theta_{\phi^{-1}(C_\varepsilon)}(\varphi(\xi_\varepsilon)) = \theta_{C_\varepsilon}(\xi_\varepsilon) = 0$$

by Lemma 1.4.9 and in turn establish the desired assertion by Theorem 1.1.27. □

PROPOSITION 1.4.11. There exists a natural bijection $(1 + \mathfrak{m}_F^*)/\mathbb{Q}_p^\times \xrightarrow{\sim} Y/\phi^{\mathbb{Z}}$ which maps the $\mathbb{Q}_p^\times$-orbit of an element $\varepsilon \in 1 + \mathfrak{m}_F^*$ to the ϕ-orbit of $y_\varepsilon \in Y$.

PROOF. The assertion is evident by Proposition 1.4.7 and Proposition 1.4.10. □

PROPOSITION 1.4.12. Every nonzero $f \in B^{\varphi = p^n}$ with $n \geq 0$ satisfies the equality

$$\mathrm{ord}_y(f) = \mathrm{ord}_{\phi(y)}(f) \quad \text{ for each } y \in Y.$$

PROOF. Choose a representative C of y. By Proposition 1.1.25 and Proposition 1.3.17, there exists a primitive element $\xi \in A_{\mathrm{inf}}$ which generates $\ker(\widehat{\theta_C})$. Proposition 1.1.10 and Lemma 1.4.9 together show that $\varphi(\xi) \in A_{\mathrm{inf}}$ is primitive and lies in $\ker(\widehat{\theta_{\phi(C)}})$. Let us write $i := \mathrm{ord}_y(f)$ and $j := \mathrm{ord}_{\phi(y)}(f)$. By Proposition 1.3.20, we may write

$$f = \xi^i g = \varphi(\xi)^j h \quad \text{ with } g, h \in B.$$

We obtain the equalities

$$f = p^{-n}\varphi(f) = \varphi(\xi)^i p^{-n}\varphi(g) \quad \text{ and } \quad f = \varphi^{-1}(\varphi(f)) = p^n\varphi^{-1}(f) = \xi^j p^n \varphi^{-1}(h),$$

which respectively yield the inequalities $i \leq j$ and $i \geq j$. Hence we find $i = j$ as desired. □

PROPOSITION 1.4.13. Every $\varepsilon \in 1 + \mathfrak{m}_F^*$ yields the identity

$$\mathrm{Div}_Y(\log(\varepsilon)) = \sum_{n \in \mathbb{Z}} \phi^n(y_\varepsilon).$$

PROOF. Proposition 1.4.5 and Proposition 1.4.12 together yield the equality

$$\mathrm{ord}_{\phi^n(y_\varepsilon)}(\log(\varepsilon)) = 1 \quad \text{ for each } n \in \mathbb{Z}.$$

Hence we only need to show that $\log(\varepsilon)$ does not vanish outside the ϕ-orbit of y_ε. Let us take an arbitrary element $y \in Y$ at which $\log(\varepsilon)$ vanishes and choose a representative C of y. Proposition 3.2.20 in Chapter II shows that $\ker(\log_{\mu_{p^\infty}})$ is the torsion subgroup of $1 + \mathfrak{m}_C$, where $\mathfrak{m}_C$ denotes the maximal ideal of $\mathcal{O}_C$. We apply Proposition 1.4.3 to find an integer n with $(\varepsilon^{p^n})^\sharp = 1$ and $(\varepsilon^{p^{n-1}})^\sharp \neq 1$. Now we have

$$\theta_C(\xi_{\varepsilon^{p^n}}) = \frac{\theta_C([\varepsilon^{p^n}] - 1)}{\theta_C([\varepsilon^{p^{n-1}}] - 1)} = \frac{(\varepsilon^{p^n})^\sharp - 1}{(\varepsilon^{p^{n-1}})^\sharp - 1} = 0$$

and thus obtain the identity $y = \phi^n(y_\varepsilon)$ by Proposition 1.4.10. □

In order to study closed points on the Fargues-Fontain curve, we invoke the following technical result without a proof.

PROPOSITION 1.4.14. A nonzero element $f \in B$ divides another nonzero element $g \in B$ if and only if we have $\mathrm{ord}_y(f) \leq \mathrm{ord}_y(g)$ for every $y \in Y$.

Remark. Proposition 1.4.14 is one of the most difficult results from the original work of Fargues-Fontaine [**FF18**]. Its proof makes heavy use of the Legendre-Newton polygons and also introduces a complete ultrametric on the set $\widehat{Y} := Y \cup \{ o \}$, where o denotes the equivalence class of the trivial untilt of F. Curious readers can find a complete proof in the article of Fargues-Fontaine [**FF18**, Chapitre 2] or the notes of Lurie [**Lur**, Lectures 13-16]. Here we state two interesting facts about the Legendre-Newton polygons used in the proof.

(1) An element $h \in B$ is a unit if and only if $\mathcal{L}_h$ is linear.
(2) Every $h \in B$ with $\partial_- \mathcal{L}_h(s) \neq \partial_+ \mathcal{L}_h(s)$ for some $s \in (0, \infty)$ must vanish at some element $y \in Y_{[p^{-s}, p^{-s}]}$.

LEMMA 1.4.15. Every $f \in B^{\varphi = p^n}$ with $n \geq 1$ vanishes at some element in Y.

PROOF. If f does not vanish at any element in Y, we deduce from Proposition 1.4.14 that f admits a multiplicative inverse in $B^{\varphi = p^{-n}}$, which contradicts Proposition 1.3.12. Hence we obtain the desired assertion. □

LEMMA 1.4.16. Let f be an element in $B^{\varphi = p^n}$ with $n \geq 1$ and ε be an element in $1 + \mathfrak{m}_F^*$. If both f and $\log(\varepsilon)$ vanish at some $y \in Y$, there exists some $g \in B^{\varphi = p^{n-1}}$ with $f = \log(\varepsilon) g$.

PROOF. By Proposition 1.4.12, we have

$$\mathrm{ord}_{\phi^i(y)}(f) = \mathrm{ord}_y(f) \geq 1 \quad \text{ for each } i \in \mathbb{Z}.$$

Hence Proposition 1.4.13 and Proposition 1.4.14 yield an element $g \in B$ with $f = \log(\varepsilon) g$. Since B is an integral domain by Proposition 1.3.14, we deduce that g lies in $B^{\varphi = p^{n-1}}$ and in turn establish the desired assertion. □

PROPOSITION 1.4.17. The map $\log : 1 + \mathfrak{m}_F \to B^{\varphi = p}$ is a continuous $\mathbb{Q}_p$-linear isomorphism.

PROOF. Choose an untilt C of F in characteristic 0. The sharp map associated to C is continuous on $\mathcal{O}_F$ by Proposition 2.1.8 in Chapter III. In addition, the map $\log_{\mu_{p^\infty}}$ is continuous by Proposition 3.2.20 in Chapter II. Since both $\widehat{\theta_C}$ and φ are continuous and open by construction, we deduce from Proposition 1.4.3 that log is continuous.

Let us now consider an arbitrary element $c \in \mathbb{Q}_p$. We may write $c = m/p^d$ with $m \in \mathbb{Z}_p$ and $d \in \mathbb{Z}$. Choose a sequence (m_i) in $\mathbb{Z}$ which converges to m under the p-adic norm. For every $\varepsilon \in 1 + \mathfrak{m}_F$, we apply Proposition 1.4.1 to find

$$\log(\varepsilon^c) = \log(\varepsilon^{m/p^d}) = \lim_{i \to \infty} \log(\varepsilon^{m_i/p^d}) = \lim_{i \to \infty} \frac{m_i}{p^d} \log(\varepsilon) = c \log(\varepsilon).$$

Hence we deduce that log is $\mathbb{Q}_p$-linear.

It remains to prove that log is an isomorphism. Proposition 1.4.13 shows that $\log(\varepsilon)$ is nonzero for every $\varepsilon \in 1 + \mathfrak{m}_F^*$ and in turn implies that log is injective. Now we only need to establish the surjectivity of log. Take an arbitrary element $f \in B^{\varphi = p}$. Proposition 1.4.7 and Lemma 1.4.15 together imply that f vanishes at $y_\varepsilon \in Y$ for some $\varepsilon \in 1 + \mathfrak{m}_F^*$. Since $\log(\varepsilon)$ also vanishes at $y_\varepsilon \in Y$ by Proposition 1.4.5, we apply Proposition 1.3.13 and Lemma 1.4.16 to obtain an element $g \in B^{\varphi = 1} \cong \mathbb{Q}_p$ with $f = \log(\varepsilon) g = \log(\varepsilon^g)$. Hence we deduce that log is surjective as desired. □

PROPOSITION 1.4.18. For every $\varepsilon \in 1 + \mathfrak{m}_F$, the element $\log(\varepsilon) \in B^{\varphi=p}$ is a prime in P.

PROOF. Proposition 1.3.14 shows that B and P are integral domains, which in particular implies that $\log(1) = 0$ is a prime in P. Let us henceforth assume that ε lies in $1 + \mathfrak{m}_F^*$. Take arbitrary elements $f, g \in P$ with fg divisible by $\log(\varepsilon)$. We wish to show that $\log(\varepsilon)$ divides either f or g in P. Since $\log(\varepsilon)$ is homogeneous, we may assume without loss of generality that both f and g are homogeneous. We note by Proposition 1.4.5 that $\log(\varepsilon)$ vanishes at $y_\varepsilon \in Y$ and in turn find by Lemma 1.3.22 that either f or g vanishes at y_ε. Hence the desired assertion follows from Lemma 1.4.16. □

PROPOSITION 1.4.19. Let f be a nonzero element in $B^{\varphi=p^n}$ for some $n \geq 1$.

(1) The map φ uniquely extends to an automorphism φ_f on $B[1/f]$.
(2) The element f admits an expression

$$f = \log(\varepsilon_1) \cdots \log(\varepsilon_n) \quad \text{with } \varepsilon_i \in 1 + \mathfrak{m}_F^* \tag{1.13}$$

where the factors are unique up to $\mathbb{Q}_p^\times$-multiple.

PROOF. Statement (1) is straightforward to verify. Let us now consider statement (2). If we have $n = 1$, the assertion is evident by Proposition 1.4.17. Hence we may assume the inequality $n > 1$. Proposition 1.4.7 and Lemma 1.4.15 together show that f vanishes at $y_{\varepsilon_n} \in Y$ for some $\varepsilon_n \in 1 + \mathfrak{m}_F^*$. Since $\log(\varepsilon_n)$ also vanishes at y_{ε_n} by Proposition 1.4.5, we apply Lemma 1.4.16 to obtain an element $g \in B^{\varphi=p^{n-1}}$ with $f = \log(\varepsilon_n)g$. Now a simple induction yields the desired expression (1.13) where the factors are unique up to $\mathbb{Q}_p^\times$-multiple by Proposition 1.3.13 and Proposition 1.4.18. □

Definition 1.4.20. For every nonzero $f \in B^{\varphi=p^n}$ with $n \geq 1$, we refer to the map φ_f in Proposition 1.4.19 as the *Frobenius automorphism* on $B[1/f]$ and often write $\varphi = \varphi_f$.

PROPOSITION 1.4.21. Let x be a nongeneric point on X.

(1) The point x is closed and corresponds to a prime $\log(\varepsilon) \in P$ for some $\varepsilon \in 1 + \mathfrak{m}_F^*$.
(2) The residue field of x is naturally isomorphic to the representatives of every $y \in Y$ at which $\log(\varepsilon)$ vanishes.

PROOF. By Proposition 1.4.19, there exists a nonzero element $t \in B^{\varphi=p}$ such that x lies in the standard open subscheme $\operatorname{Spec}(B[1/t]^{\varphi=1})$ of $X = \operatorname{Proj}(P)$. Let us denote by $\mathfrak{p}$ the prime ideal of $B[1/t]^{\varphi=1}$ which corresponds to x. If we take a nonzero element $f \in B^{\varphi=p^n}$ for some $n \geq 1$ with $f/t^n \in \mathfrak{p}$, we use Proposition 1.4.19 to find

$$\frac{f}{t^n} = \frac{\log(\varepsilon_1)}{t} \cdot \frac{\log(\varepsilon_2)}{t} \cdots \frac{\log(\varepsilon_n)}{t} \quad \text{with } \varepsilon_i \in 1 + \mathfrak{m}_F^*.$$

and thus obtain an element $\varepsilon \in 1 + \mathfrak{m}_F^*$ with $\log(\varepsilon)/t \in \mathfrak{p}$.

Consider an element $y \in Y$ at which $\log(\varepsilon)$ vanishes and choose a representative C of y. If t vanishes at y, we see by Proposition 1.3.13 and Lemma 1.4.16 that $\log(\varepsilon)/t$ is invertible for being in $B^{\varphi=1} \cong \mathbb{Q}_p$, which is impossible as $\mathfrak{p}$ is a prime ideal. Hence we deduce that t does not vanish at y and in turn obtain a map $\theta_x : B[1/t]^{\varphi=1} \to C$ induced by $\widehat{\theta_C}$.

Proposition 1.4.3 shows that $\widehat{\theta_C}$ restricts to a surjective map $B^{\varphi=p} \twoheadrightarrow C$, which in particular implies that θ_x is surjective. Moreover, given an element $g \in B^{\varphi=p^n}$ for some $n \geq 1$ with $g/t^n \in \ker(\theta_x)$, we note that g vanishes at y and accordingly find by Lemma 1.4.16 that $\log(\varepsilon)/t$ divides g/t^n. Since $\log(\varepsilon)/t$ lies in $\ker(\theta_x)$, we see that $\log(\varepsilon)/t$ generates $\ker(\theta_x)$ and thus deduce that $\mathfrak{p}$ coincides with the maximal ideal $\ker(\theta_x)$ in $B[1/t]^{\varphi=1}$. Now the desired assertions are evident. □

THEOREM 1.4.22 (Fargues-Fontaine [**FF18**]). Let $|X|$ denote the set of closed points on X.

(1) There exists a natural bijection $|X| \xrightarrow{\sim} Y/\phi^{\mathbb{Z}}$ which maps the point on X given by a prime $\log(\varepsilon) \in P$ for some $\varepsilon \in 1 + \mathfrak{m}_F^*$ to the ϕ-orbit of $y_\varepsilon \in Y$.

(2) X is a Dedekind scheme such that the open subscheme $X \setminus \{ x \}$ for every $x \in |X|$ is the spectrum of a principal ideal domain.

(3) For every $x \in |X|$, its completed local ring $\widehat{\mathcal{O}_{X,x}}$ admits a natural identification
$$\widehat{\mathcal{O}_{X,x}} \cong B_{\mathrm{dR}}^+(y)$$
where y is an arbitrary element in the image of x under the bijection $|X| \xrightarrow{\sim} Y/\phi^{\mathbb{Z}}$.

PROOF. Proposition 1.4.21 yields a surjection $1 + \mathfrak{m}_F^* \twoheadrightarrow |X|$ sending each $\varepsilon \in 1 + \mathfrak{m}_F^*$ to the point on X given by the prime $\log(\varepsilon) \in P$. By Proposition 1.3.13 and Proposition 1.4.17, two elements ε_1 and ε_2 in $1 + \mathfrak{m}_F^*$ map to the same point on X if and only if ε_1 and ε_2 lie in the same $\mathbb{Q}_p^\times$-orbit. Therefore we deduce statement (1) from Proposition 1.4.11.

Let us now fix a closed point x on X which corresponds to the the prime $\log(\varepsilon) \in P$ for some $\varepsilon \in 1 + \mathfrak{m}_F^*$. The scheme $X \setminus \{ x \}$ is naturally isomorphic to $\mathrm{Spec}\,(B[1/\log(\varepsilon)]^{\varphi=1})$. In addition, Proposition 1.4.21 shows that every prime ideal of $B[1/\log(\varepsilon)]^{\varphi=1}$ is principal. Hence we obtain statement (2) by a general fact stated in the Stacks project [**Sta**, Tag 05KH].

It remains to establish statement (3). Let us take an element $y \in Y$ in the ϕ-orbit of y_ε and choose a representative C of y. Proposition 1.4.19 yields a nonzero element $t \in B^{\varphi=p}$ such that x lies in the open subscheme $\mathrm{Spec}\,(B[1/t]^{\varphi=1})$ of X. We see by Proposition 1.4.21 that x corresponds to the maximal ideal $\mathfrak{m}_x$ of $B[1/t]^{\varphi=1}$ generated by $\log(\varepsilon)/t$ and in turn get a natural isomorphism
$$\widehat{\mathcal{O}_{X,x}} \cong \varprojlim_i B[1/t]^{\varphi=1}/\mathfrak{m}_x^i.$$
Meanwhile, since t is not a $\mathbb{Q}_p^\times$-multiple of $\log(\varepsilon)$, Proposition 1.3.13 and Lemma 1.4.16 together show that t does not vanish at y. Let $\widehat{\theta_C}[1/t] : B[1/t] \twoheadrightarrow C$ denote the surjective ring homomorphism induced by $\widehat{\theta_C}$. We apply Proposition 1.3.20 to identify $B_{\mathrm{dR}}^+(y)$ with the completed local ring of the closed point on $\mathrm{Spec}\,(B)$ given by $\ker(\widehat{\theta_C})$ and thus obtain a canonical isomorphism
$$B_{\mathrm{dR}}^+(y) \cong \varprojlim_i B[1/t]/\ker(\widehat{\theta_C}[1/t])^i.$$
If we consider an integer $i \geq 1$ and an element $f \in B^{\varphi=p^n}$ for some $n \geq 1$ such that f/t^n lies in $B[1/t]^{\varphi=1} \cap \ker(\widehat{\theta_C}[1/t])^i$, we find $\mathrm{ord}_y(f) \geq i$ and in turn deduce from Lemma 1.4.16 that $\log(\varepsilon)^i/t^i$ divides f/t^n. Hence the ideal $\mathfrak{m}_x$ generated by $\log(\varepsilon)/t$ admits an identification
$$\mathfrak{m}_x^i = B[1/t]^{\varphi=1} \cap \ker(\widehat{\theta_C}[1/t])^i \quad \text{ for each } i \geq 1.$$
Now we obtain a natural injective ring homomorphism
$$\widehat{\mathcal{O}_{X,x}} \cong \varprojlim_i B[1/t]^{\varphi=1}/\mathfrak{m}_x^i \longrightarrow \varprojlim_i B[1/t]/\ker(\widehat{\theta_C}[1/t])^i \cong B_{\mathrm{dR}}^+(y).$$
Moreover, since both $B[1/t]^{\varphi=1}/\mathfrak{m}_x$ and $B[1/t]/\ker(\widehat{\theta_C}[1/t])$ are isomorphic to C, this map is surjective by a general fact stated in the Stacks project [**Sta**, Tag 0315]. Therefore we establish the desired assertion. □

Remark. Theorem 1.4.22 shows notable similarities between the Fargues-Fontaine curve X and the complex projective line $\mathbb{P}_{\mathbb{C}}^1$, although X is not of finite type over the base field $\mathbb{Q}_p$. In the subsequent section, we will present many additional similarities between X and $\mathbb{P}_{\mathbb{C}}^1$.

2. Vector bundles

Our main objective in this section is to discuss several key properties of vector bundles on the Fargues-Fontaine curve. The primary references for this section are the survey article of Fargues-Fontaine [**FF12**] and the lecture notes of Lurie [**Lur**].

2.1. Line bundles and their cohomology

Throughout this subsection, we denote by $|X|$ the set of closed points on X.

LEMMA 2.1.1. The group of Weil divisors on X is the free abelian group generated by $|X|$.

PROOF. The assertion is an immediate consequence of Theorem 1.4.22. □

Definition 2.1.2. The *divisor degree map* of X is the homomorphism $\deg : \mathrm{Div}(X) \to \mathbb{Z}$ with $\deg(x) = 1$ for every $x \in |X|$, where $\mathrm{Div}(X)$ denotes the group of Weil divisors on X.

LEMMA 2.1.3. There exists a natural bijection $|X| \xrightarrow{\sim} (B^{\varphi=p}\setminus \{ 0 \})/\mathbb{Q}_p^\times$ which maps the point on X given by a prime $\log(\varepsilon) \in P$ for some $\varepsilon \in 1+\mathfrak{m}_F^*$ to the $\mathbb{Q}_p^\times$-orbit of $\log(\varepsilon) \in B^{\varphi=p}$.

PROOF. The assertion is evident by Proposition 1.4.17 and Theorem 1.4.22. □

PROPOSITION 2.1.4. A Weil divisor D on X is principal if and only if we have $\deg(D) = 0$.

PROOF. If D is a principal divisor of a rational function f on X, we have $f = g/h$ for some $g, h \in B^{\varphi=p^n}$ with $n \geq 0$ and thus apply Proposition 1.4.19 to obtain the identity

$$f = \frac{t_1 t_2 \cdots t_n}{t_{n+1}t_{n+2}\cdots t_{2n}} \qquad \text{with } t_i \in B^{\varphi=p},$$

which in turn yields the equality $\deg(D) = 0$ by Lemma 2.1.3. Conversely, if D satisfies the equality $\deg(D) = 0$, we write

$$D = (x_1 + x_2 + \cdots + x_n) - (x_{n+1} + x_{n+2} + \cdots + x_{2n}) \qquad \text{with } x_i \in |X|$$

and use Lemma 2.1.3 to get a rational function f on X whose principal Weil divisor is D. □

Definition 2.1.5. Given an integer d, the *d-fold Serre twist of $\mathcal{O}_X$* is the quasicoherent $\mathcal{O}_X$-module $\mathcal{O}(d) = \mathcal{O}_X(d)$ associated to $P(d) := \bigoplus_{n\geq 0} B^{\varphi=p^{d+n}}$.

Remark. For $\mathbb{P}^1_{\mathbb{C}} = \mathrm{Proj}\,(\mathbb{C}[z_1, z_2])$, we can similarly define the Serre twist $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d)$ of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}$.

PROPOSITION 2.1.6. The divisor degree map of X induces a natural isomorphism $\mathrm{Pic}(X) \cong \mathbb{Z}$ whose inverse maps each $d \in \mathbb{Z}$ to the isomorphism class of $\mathcal{O}(d)$.

PROOF. We may identify $\mathrm{Pic}(X)$ with the class group of X by Theorem 1.4.22 and a general fact stated in the Stacks project [**Sta**, Tag 0BE9]. Hence Proposition 2.1.4 shows that the divisor degree map of X induces a natural isomorphism $\mathrm{Pic}(X) \cong \mathbb{Z}$. Let us now consider an arbitrary integer d. Since the elements in $B^{\varphi=p}$ generate the $\mathbb{Q}_p$-algebra P as noted in Proposition 1.4.19, the $\mathcal{O}_X$-module $\mathcal{O}(d)$ is a line bundle on X by a general fact stated in the Stacks project [**Sta**, Tag 01MT]. Take a nonzero element $t \in B^{\varphi=p}$, which induces a closed point x on X by Lemma 2.1.3. We observe that t^d yields a global section of $\mathcal{O}(d)$ and in turn find that $\mathcal{O}(d)$ is isomorphic to the line bundle given by the Weil divisor dx on X. Hence the isomorphism class of $\mathcal{O}(d)$ maps to d under the isormorphism $\mathrm{Pic}(X) \cong \mathbb{Z}$. □

Remark. Similarly, $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}})$ admits a natural isomorphism $\mathrm{Pic}(\mathbb{P}^1_{\mathbb{C}}) \cong \mathbb{Z}$ whose inverse maps each $d \in \mathbb{Z}$ to the isomorphism class of $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d)$.

PROPOSITION 2.1.7. Let d be a nonnegative integer and t be a nonzero element in $B^{\varphi=p}$. Denote by $t\mathcal{O}(d)$ the quasicoherent $\mathcal{O}_X$-module associated to $tP(d)$.

(1) There exists a natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^{\varphi=p^d} & \longrightarrow & B^{\varphi=p^{d+1}} & \longrightarrow & B^{\varphi=p^{d+1}}/tB^{\varphi=p^d} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \wr\downarrow & & \\ 0 & \longrightarrow & H^0(X, \mathcal{O}(d)) & \longrightarrow & H^0(X, \mathcal{O}(d+1)) & \longrightarrow & H^0(X, \mathcal{O}(d+1)/t\mathcal{O}(d)) & \longrightarrow & 0 \end{array}$$

where both rows are exact.

(2) The $\mathcal{O}_X$-module $\mathcal{O}(d+1)/t\mathcal{O}(d)$ is a skyscraper sheaf at $x \in |X|$ induced by t.

PROOF. Since P is an integral domain by Proposition 1.3.14, the multiplication by t on P induces an exact sequence

$$0 \longrightarrow P(d) \longrightarrow P(d+1) \longrightarrow P(d+1)/tP(d) \longrightarrow 0 \tag{2.1}$$

which gives rise to an exact sequence of coherent $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{O}(d) \longrightarrow \mathcal{O}(d+1) \longrightarrow \mathcal{O}(d+1)/t\mathcal{O}(d) \longrightarrow 0. \tag{2.2}$$

The top row in the diagram comes from the sequence (2.1) and is exact. The bottom row in the diagram comes from the sequence (2.2) and is left exact. Moreover, a general fact stated in the Stacks project [**Sta**, Tag 01M7] yields the vertical maps in the diagram and subsequently establishes the commutativity of the diagram.

Meanwhile, Lemma 1.4.15 yields an element $y \in Y$ at which t vanishes. Let us choose a representative C of y. Proposition 1.4.3 shows that $\widehat{\theta_C}$ restricts to a surjective map $B^{\varphi=p} \twoheadrightarrow C$. We see that $\widehat{\theta_C}$ also restricts to a surjective map $B^{\varphi=p^{d+1}} \twoheadrightarrow C$; indeed, for every $a \in C$, we take $f, g \in B^{\varphi=p}$ with $\widehat{\theta_C}(f) = 1$ and $\widehat{\theta_C}(g) = a$ to find $\widehat{\theta_C}(f^d g) = a$. In addition, we find by Lemma 1.4.16 that the kernel of the surjective map $B^{\varphi=p^{d+1}} \twoheadrightarrow C$ is $tB^{\varphi=p^d}$. Therefore the map $\widehat{\theta_C}$ gives rise to an isomorphism

$$B^{\varphi=p^{d+1}}/tB^{\varphi=p^d} \xrightarrow{\sim} C. \tag{2.3}$$

Now we apply Lemma 2.1.3 to take $x \in |X|$ induced by t and use Proposition 1.4.21 to identify C with the residue field of x. Proposition 2.1.6 implies that $\mathcal{O}(d)$ and $\mathcal{O}(d+1)$ are respectively isomorphic to the line bundles given by the Weil divisors dx and $(d+1)x$ on X. Hence the injective $\mathcal{O}_X$-module morphism $\mathcal{O}(d) \hookrightarrow \mathcal{O}(d+1)$ in the sequence (2.2) induces an isomorphism on the stalks at every $x' \in |X|$ with $x' \neq x$. We see that $\mathcal{O}(d+1)/t\mathcal{O}(d)$ is isomorphic to the skyscraper sheaf at x with value $t^{-d-1}\mathcal{O}_{X,x}/t^{-d}\mathcal{O}_{X,x} \simeq C$ and in turn use the isomorphism (2.3) to find that the right vertical map in the diagram is an isomorphism. Moreover, we obtain the exactness of the bottom row in the diagram by the commutativity of the right square. Therefore we establish the desired assertion. □

Remark. For $d = 0$, our proof of Proposition 2.1.7 yields a short exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B^{\varphi=p} \longrightarrow C \longrightarrow 0.$$

In fact, the work of Colmez [**Col02**] shows that many key objects in p-adic Hodge theory arise as extensions of a finite dimensional C-vector space by a finite dimensional $\mathbb{Q}_p$-vector space, referred to as *Banach-Colmez spaces*. Moreover, the result of le Bras [**LB18**] presents a classificaiton of Banach-Colmez spaces in terms of coherent $\mathcal{O}_X$-modules. We refer curious readers to the book of Scholze-Weinstein [**SW20**, §15.2] for details.

THEOREM 2.1.8 (Fargues-Fontaine [**FF18**]). For the cohomology of line bundles on X, we have the following statements:

(1) There exists a canonical isomorphism $H^0(X, \mathcal{O}(d)) \cong B^{\varphi=p^d}$ for every $d \in \mathbb{Z}$.
(2) The cohomology group $H^1(X, \mathcal{O}(d))$ vanishes for every $d \geq 0$.

PROOF. Let us take a nonzero element $t \in B^{\varphi=p}$. By Lemma 2.1.3, there exists a closed point x on X induced by t. We note that the scheme $U := X \setminus \{ x \}$ admits a natural isomorphism $U \cong \operatorname{Spec}(B[1/t]^{\varphi=1})$.

Since P is an integral domain by Proposition 1.3.14, for every $d \in \mathbb{Z}$ the multiplication by t on P yields an injective map $P(d) \hookrightarrow P(d+1)$ and in turn induces an injective $\mathcal{O}_X$-module morphism $\mathcal{O}(d) \hookrightarrow \mathcal{O}(d+1)$. Meanwhile, Proposition 2.1.6 shows that each $\mathcal{O}(d)$ is isomorphic to the line bundle given by the Weil divisor dx on X. We see that $\varinjlim \mathcal{O}(d)$ is naturally isomorphic to the pushforward of $\mathcal{O}_U$ via the embedding $U \hookrightarrow X$ and thus obtain natural isomorphisms

$$H^0\left(X, \varinjlim \mathcal{O}(d)\right) \cong H^0(U, \mathcal{O}_U) \cong B[1/t]^{\varphi=1}, \tag{2.4}$$

$$H^1\left(X, \varinjlim \mathcal{O}(d)\right) \cong H^1(U, \mathcal{O}_U) = 0. \tag{2.5}$$

Let us now prove statement (1). For every $d \in \mathbb{Z}$, Proposition 2.1.7 yields a natural homomorphism $\alpha_d : B^{\varphi=p^d} \to H^0(X, \mathcal{O}(d))$. We wish to show that each α_d is an isomorphism. The sequence (α_d) gives rise to a homomorphism

$$B[1/t]^{\varphi=1} \cong \varinjlim B^{\varphi=p^d} \longrightarrow \varinjlim H^0(X, \mathcal{O}(d)) \cong H^0\left(X, \varinjlim \mathcal{O}(d)\right).$$

It is straightforward to verify that this map coincides with the isomorphism (2.4). Moreover, Proposition 2.1.7 and the snake lemma together yield isomorphisms

$$\ker(\alpha_d) \simeq \ker(\alpha_{d+1}) \quad \text{and} \quad \operatorname{coker}(\alpha_d) \simeq \operatorname{coker}(\alpha_{d+1}) \quad \text{for each } d \geq 0.$$

Therefore α_d is an isomorphism for each $d \geq 0$. Now by Proposition 1.3.13 we have

$$H^0(X, \mathcal{O}_X) \cong B^{\varphi=1} \cong \mathbb{Q}_p.$$

For each $d < 0$, we see that $H^0(X, \mathcal{O}_X)$ does not contain a nonzero element with vanishing order $-d$ at x, which means that $H^0(X, \mathcal{O}(d))$ is trivial. Hence Proposition 1.3.12 shows that α_d is an isomorphism for each $d < 0$ as well.

It remains to establish statement (2). For every $d \geq 0$, Proposition 2.1.7 implies that the cohomology group $H^1(X, \mathcal{O}(d+1)/t\mathcal{O}(d))$ vanishes and in turn yields a long exact sequence

$$H^0(X, \mathcal{O}(d+1)) \longrightarrow H^0(X, \mathcal{O}(d+1)/t\mathcal{O}(d)) \longrightarrow H^1(X, \mathcal{O}(d)) \longrightarrow H^1(X, \mathcal{O}(d+1)) \longrightarrow 0.$$

where the first map is surjective. Now we find

$$H^1(X, \mathcal{O}(d)) \simeq H^1(X, \mathcal{O}(d+1)) \quad \text{for each } d \geq 0$$

and thus use the isomorphism (2.5) to establish the desired assertion. □

Remark. Theorem 2.1.8 provides analogues of the following facts about $\mathbb{P}^1_{\mathbb{C}}$:

(1) For every $d \in \mathbb{Z}$, the cohomology group $H^0(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d))$ is naturally isomorphic to the group of homogeneous polynomials of degree d in $\mathbb{C}[z_1, z_2]$.
(2) For every $d \geq 0$, the cohomology group $H^1(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d))$ vanishes.

2.2. Harder-Narasimhan filtration

In this subsection, we introduce a general formalism for studying vector bundles on algebraic curves and similar objects.

Definition 2.2.1. A *complete abstract curve* is a scheme Z with the following properties:

(i) Z is connected, separated, noetherian and regular of dimension 1.

(ii) Z admits a homomorphism $\deg_Z : \mathrm{Pic}(Z) \to \mathbb{Z}$, called a *degree map*, which takes a positive value on every line bundle given by a nonzero effective Weil divisor on Z.

Example 2.2.2. Below are two important examples of complete abstract curves.

(1) Every regular proper curve over a field is a complete abstract curve by a general fact stated in the Stacks project [**Sta**, Tag 0AYY].

(2) The Fargues-Fontaine curve is a complete abstract curve by Theorem 1.4.22 and Proposition 2.1.6.

PROPOSITION 2.2.3. Let Z be a complete abstract curve.

(1) The scheme Z is integral.

(2) Every line bundle on Z given by a principal Weil divisor maps to 0 under $\deg_Z$.

(3) The cohomology group $H^0(Z, \mathcal{O}_Z)$ is naturally a field.

PROOF. The first two statements are consequences of standard facts stated in the Stacks project [**Sta**, Tag 033N and Tag 0BE9]. For the last statement, let us denote the function field of Z by $K(Z)$ and the Weil divisor of an element $f \in K(Z)^\times$ by $\mathrm{Div}_Z(f)$. The cohomology group $H^0(Z, \mathcal{O}_Z)$ is naturally a subring of $K(Z)$ via the identification $H^0(Z, \mathcal{O}_Z) \cong \mathcal{O}_Z(Z)$; indeed, an element $f \in K(Z)^\times$ yields a global section of $\mathcal{O}_Z$ if and only if $\mathrm{Div}_Z(f)$ is effective. Meanwhile, by the second statement, a principal Weil divisor on Z is effective if and only if it is trivial. Hence we find

$$H^0(Z, \mathcal{O}_Z) \backslash \{ 0 \} = \{ f \in K(Z)^\times : \mathrm{Div}_Z(f) = 0 \}$$

and in turn identify $H^0(Z, \mathcal{O}_Z)$ with a subfield of $K(Z)$. □

PROPOSITION 2.2.4. Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on a complete abstract curve Z.

(1) If we have $\deg_Z(\mathcal{L}) > \deg_Z(\mathcal{M})$, every $\mathcal{O}_Z$-module map from $\mathcal{L}$ to $\mathcal{M}$ is zero.

(2) If we have $\deg_Z(\mathcal{L}) = \deg_Z(\mathcal{M})$, every nonzero $\mathcal{O}_Z$-module map from $\mathcal{L}$ to $\mathcal{M}$ is an isomorphism.

PROOF. Let us assume that there exists a nonzero $\mathcal{O}_Z$-module map $f : \mathcal{L} \to \mathcal{M}$. Denote the dual bundle of $\mathcal{L}$ by $\mathcal{L}^\vee$. We may identify f with a nonzero global section of $\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}$ via the identification

$$\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{L}, \mathcal{M}) \cong H^0(Z, \mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}). \tag{2.6}$$

Hence $\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}$ arises from an effective Weil divisor D on Z by a general fact stated in the Stacks project [**Sta**, Tag 01X0]. Now we find

$$\deg_Z(\mathcal{M}) - \deg_Z(\mathcal{L}) = \deg_Z(\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}) \geq 0$$

and in turn obtain statement (1).

For statement (2), we henceforth assume the equality $\deg_Z(\mathcal{L}) = \deg_Z(\mathcal{M})$. Since we have $\deg_Z(\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}) = 0$, we see that the effective Weil divisor D on Z is zero, which means that $\mathcal{L}^\vee \otimes_{\mathcal{O}_Z} \mathcal{M}$ is trivial. Hence the isomorphism (2.6) and Proposition 2.2.3 together imply that f is an isomorphism as desired. □

For the rest of this subsection, we let Z be a complete abstract curve.

Definition 2.2.5. Let $\mathcal{V}$ be a nonzero vector bundle on Z.

(1) We define the *degree* of $\mathcal{V}$ to be $\deg(\mathcal{V}) := \deg_Z(\det(\mathcal{V}))$.

(2) We write $\mathrm{rk}(\mathcal{V})$ for the rank of $\mathcal{V}$ and define the *slope* of $\mathcal{V}$ to be $\mu(\mathcal{V}) := \dfrac{\deg(\mathcal{V})}{\mathrm{rk}(\mathcal{V})}$.

Remark. If we define the degree of the zero bundle to be 0, many results about nonzero vector bundles in this subsection are valid for zero bundles.

LEMMA 2.2.6. For nonzero free modules M and N over a ring R respectively of rank r and s, there exists a natural isomorphism

$$\det(M \otimes_R N) \cong \det(M)^{\otimes s} \otimes_R \det(N)^{\otimes r}. \tag{2.7}$$

PROOF. Let us choose R-bases (m_i) and (n_j) respectively for M and N. We obtain the isomorphism (2.7) by mapping $\bigwedge(m_i \otimes n_j)$ to $(\bigwedge m_i)^{\otimes s} \otimes (\bigwedge n_j)^{\otimes r}$. This map does not depend on the choice of R-bases for M and N; indeed, if we take R-module automorphisms f and g respsectively for M and N, we obtain the equalities

$$\bigwedge (f(m_i) \otimes g(n_j)) = \det(f)^s \det(g)^r \bigwedge (m_i \otimes n_j),$$
$$\left(\bigwedge f(m_i)\right)^{\otimes s} \otimes \left(\bigwedge g(n_j)\right)^{\otimes r} = \det(f)^s \det(g)^r \left(\bigwedge m_i\right)^{\otimes s} \otimes \left(\bigwedge n_j\right)^{\otimes r}.$$

Therefore we establish the desired assertion. ☐

PROPOSITION 2.2.7. Given nonzero vector bundles $\mathcal{V}$ and $\mathcal{W}$ on Z, we have

$$\deg(\mathcal{V} \otimes_{\mathcal{O}_Z} \mathcal{W}) = \deg(\mathcal{V})\mathrm{rk}(\mathcal{W}) + \deg(\mathcal{W})\mathrm{rk}(\mathcal{V}) \quad \text{and} \quad \mu(\mathcal{V} \otimes_{\mathcal{O}_Z} \mathcal{W}) = \mu(\mathcal{V}) + \mu(\mathcal{W}).$$

PROOF. The first equality is evident by Lemma 2.2.6. The second equality follows from the first equality as we have $\mathrm{rk}(\mathcal{V} \otimes_{\mathcal{O}_Z} \mathcal{W}) = \mathrm{rk}(\mathcal{V})\mathrm{rk}(\mathcal{W})$. ☐

PROPOSITION 2.2.8. Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be nonzero vector bundles on Z with an exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0.$$

(1) $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ satisfy the equalities

$$\mathrm{rk}(\mathcal{V}) = \mathrm{rk}(\mathcal{U}) + \mathrm{rk}(\mathcal{W}) \quad \text{and} \quad \deg(\mathcal{V}) = \deg(\mathcal{U}) + \deg(\mathcal{W}).$$

(2) $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ satisfy the inequality

$$\min(\mu(\mathcal{U}), \mu(\mathcal{W})) \leq \mu(\mathcal{V}) \leq \max(\mu(\mathcal{U}), \mu(\mathcal{W}))$$

with equality precisely when $\mu(\mathcal{U})$ and $\mu(\mathcal{W})$ are equal.

PROOF. The first identity in statement (1) is evident, whereas the second identity in statement (1) follows from a general fact stated in the Stacks project [**Sta**, Tag 0B38]. Moreover, by statement (1) we have

$$\mu(\mathcal{V}) = \frac{\deg(\mathcal{V})}{\mathrm{rk}(\mathcal{V})} = \frac{\deg(\mathcal{U}) + \deg(\mathcal{W})}{\mathrm{rk}(\mathcal{U}) + \mathrm{rk}(\mathcal{W})}.$$

and thus obtain statement (2). ☐

Remark. We can define the degree of an arbitrary nonzero coherent $\mathcal{O}_Z$-module such that Proposition 2.2.8 extends to nonzero coherent $\mathcal{O}_Z$-modules $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ with an exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0.$$

PROPOSITION 2.2.9. Let $\mathcal{V}$ be a vector bundle on Z and $\mathcal{W}$ be its coherent $\mathcal{O}_Z$-submodule.

(1) $\mathcal{W}$ is a vector bundle on Z.

(2) If $\mathcal{W}$ is nonzero, there exists a subbundle $\widetilde{\mathcal{W}}$ of $\mathcal{V}$ with the following properties:

 (i) $\widetilde{\mathcal{W}}$ contains $\mathcal{W}$ as a coherent $\mathcal{O}_Z$-submodule and satisfies the relations

$$\mathrm{rk}(\mathcal{W}) = \mathrm{rk}(\widetilde{\mathcal{W}}), \quad \deg(\mathcal{W}) \leq \deg(\widetilde{\mathcal{W}}).$$

 (ii) $\widetilde{\mathcal{W}}$ coincides with $\mathcal{W}$ if and only if it satisfies the equality $\deg(\mathcal{W}) = \deg(\widetilde{\mathcal{W}})$.

PROOF. The scheme Z is integral as noted in Proposition 2.2.3. By a general fact stated in the Stacks project [**Sta**, Tag 0CC4], a coherent $\mathcal{O}_Z$-module is a vector bundle if and only if it is torsion free. Hence we deduce that $\mathcal{W}$ is a vector bundle on Z.

Let us henceforth assume that $\mathcal{W}$ is nonzero. We write $\mathcal{T}$ for the torsion subsheaf of the quotient $\mathcal{V}/\mathcal{W}$ and take $\widetilde{\mathcal{W}}$ to be the preimage of $\mathcal{T}$ under the natural surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{W}$. We see that $\widetilde{\mathcal{W}}$ is a subbundle of $\mathcal{V}$ as $\mathcal{V}/\widetilde{\mathcal{W}}$ is torsion free. Moreover, $\widetilde{\mathcal{W}}$ contains $\mathcal{W}$ as a coherent $\mathcal{O}_Z$-module with $\widetilde{\mathcal{W}}/\mathcal{W} \simeq \mathcal{T}$ being a torsion sheaf. Hence we find $\mathrm{rk}(\widetilde{\mathcal{W}}) = \mathrm{rk}(\mathcal{W})$ and in turn obtain a nonzero $\mathcal{O}_Z$-module homomorphism $f : \det(\mathcal{W}) \to \det(\widetilde{\mathcal{W}})$ induced by the embedding $\mathcal{W} \hookrightarrow \widetilde{\mathcal{W}}$. Now Proposition 2.2.4 yields the inequality $\deg(\mathcal{W}) \leq \deg(\widetilde{\mathcal{W}})$. In addition, if we have $\deg(\mathcal{W}) = \deg(\widetilde{\mathcal{W}})$, the embedding $\mathcal{W} \hookrightarrow \mathcal{V}$ is an isomorphism as its determinant f is an isomorphism by Proposition 2.2.4; indeed, since the induced maps on the stalks are injective, they are isomorphisms precisely when the determinant is invertible. Therefore we establish the desired assertions. □

Remark. In general, $\mathcal{W}$ is not necessarily a subbundle of $\mathcal{V}$ as the quotient $\mathcal{V}/\mathcal{W}$ may have a nonzero torsion subsheaf.

Definition 2.2.10. For a vector bundle $\mathcal{V}$ over Z with a nonzero coherent $\mathcal{O}_Z$-submodule $\mathcal{W}$, we refer to the vector bundle $\widetilde{\mathcal{W}}$ on Z given by Proposition 2.2.9 as the *saturation* of $\mathcal{W}$ in $\mathcal{V}$.

PROPOSITION 2.2.11. Given a nonzero vector bundle $\mathcal{V}$ on Z, there exists an integer $d_{\mathcal{V}}$ with $\deg(\mathcal{W}) \leq d_{\mathcal{V}}$ for every nonzero subbundle $\mathcal{W}$ of $\mathcal{V}$.

PROOF. If $\mathcal{V}$ is a line bundle, we obtain the desired assertion with $d_{\mathcal{V}} = \deg(\mathcal{V})$ as $\mathcal{V}$ is its only nonzero subbundle. Let us now assume the inequality $\mathrm{rk}(\mathcal{V}) > 1$ and proceed by induction on $\mathrm{rk}(\mathcal{V})$. If $\mathcal{V}$ is its only nonzero subbundle, the assertion is evident. Hence we may also assume that there exists a nonzero subbundle $\mathcal{U}$ of $\mathcal{V}$ with $\mathcal{U} \neq \mathcal{V}$. Consider an arbitrary nonzero subbundle $\mathcal{W}$ of $\mathcal{V}$. We write $\mathcal{P} := \mathcal{W} \cap \mathcal{U}$ and denote by $\mathcal{Q}$ the image of $\mathcal{W}$ under the natural surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{U}$. Proposition 2.2.9 shows that $\mathcal{P}$ and $\mathcal{Q}$ are vector bundles on Z. Hence by the induction hypothesis, we have

$$\deg(\mathcal{P}) \leq d_{\mathcal{U}} \quad \text{and} \quad \deg(\mathcal{Q}) \leq d_{\mathcal{V}/\mathcal{U}}$$

for some integers $d_{\mathcal{U}}$ and $d_{\mathcal{V}/\mathcal{U}}$ which do not depend on $\mathcal{W}$. Since we have an exact sequence

$$0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{W} \longrightarrow \mathcal{Q} \longrightarrow 0,$$

we apply Proposition 2.2.8 to find

$$\deg(\mathcal{W}) = \deg(\mathcal{P}) + \deg(\mathcal{Q}) \leq d_{\mathcal{U}} + d_{\mathcal{V}/\mathcal{U}},$$

thereby completing the proof. □

Remark. On the other hand, unless $\mathcal{V}$ has rank 1, we don't necessarily have an integer $d'_{\mathcal{V}}$ with $\deg(\mathcal{W}) \geq d'_{\mathcal{V}}$ for every subbundle $\mathcal{W}$ of $\mathcal{V}$.

Definition 2.2.12. Let $\mathcal{V}$ be a nonzero vector bundle on Z.

(1) $\mathcal{V}$ is *semistable* if we have $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$ for every nonzero subbundle $\mathcal{W}$ of $\mathcal{V}$.
(2) $\mathcal{V}$ is *stable* if we have $\mu(\mathcal{W}) < \mu(\mathcal{V})$ for every nonzero subbundle $\mathcal{W}$ of $\mathcal{V}$ with $\mathcal{W} \neq \mathcal{V}$.

Example 2.2.13. Every line bundle on Z is stable as it is its only nonzero subbundle.

PROPOSITION 2.2.14. Given a semistable vector bundle $\mathcal{V}$ on Z, every nonzero coherent $\mathcal{O}_Z$-submodule $\mathcal{W}$ of $\mathcal{V}$ is a vector bundle on Z with $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$.

PROOF. Since $\mathcal{W}$ is a vector bundle on Z as noted in Proposition 2.2.9, we take the saturation $\widetilde{\mathcal{W}}$ of $\mathcal{W}$ in $\mathcal{V}$ and find $\mu(\mathcal{W}) \leq \mu(\widetilde{\mathcal{W}}) \leq \mu(\mathcal{V})$. □

PROPOSITION 2.2.15. Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be nonzero vector bundles on Z with an exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{V} \longrightarrow \mathcal{W} \longrightarrow 0.$$

(1) If $\mathcal{U}$ and $\mathcal{V}$ are semistable of slope λ, then $\mathcal{W}$ is also semistable of slope λ.
(2) If $\mathcal{V}$ and $\mathcal{W}$ are semistable of slope λ, then $\mathcal{U}$ is also semistable of slope λ.
(3) If $\mathcal{U}$ and $\mathcal{W}$ are semistable of slope λ, then $\mathcal{V}$ is also semistable of slope λ.

PROOF. Let us first assume for statement (1) that $\mathcal{U}$ and $\mathcal{V}$ are semistable of slope λ. Proposition 2.2.8 implies that $\mathcal{W}$ has slope λ. Take an arbitrary subbundle $\mathcal{Q}$ of $\mathcal{W}$ and denote by $\mathcal{Q}'$ the preimage of $\mathcal{Q}$ under the map $\mathcal{V} \twoheadrightarrow \mathcal{W}$. We have a short exact sequence

$$0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{Q}' \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Moreover, Proposition 2.2.14 shows that $\mathcal{Q}'$ is a vector bundle on Z with $\mu(\mathcal{Q}') \leq \mu(\mathcal{V}) = \lambda$. Hence we find $\mu(\mathcal{Q}) \leq \lambda$ by Proposition 2.2.8 and in turn see that $\mathcal{W}$ is semistable of slope λ.

We now assume for statement (2) that $\mathcal{V}$ and $\mathcal{W}$ are semistable of slope λ. Proposition 2.2.8 implies that $\mathcal{U}$ has slope λ. Since every subbundle of $\mathcal{U}$ is a coherent $\mathcal{O}_Z$-submodule of $\mathcal{V}$, we deduce from Proposition 2.2.14 that $\mathcal{U}$ is semistable of slope λ.

Finally, let us assume for statement (3) that $\mathcal{U}$ and $\mathcal{W}$ are semistable of slope λ. Proposition 2.2.8 implies $\mathcal{V}$ has slope λ. Take an arbitrary subbundle $\mathcal{R}$ of $\mathcal{V}$ and denote by $\mathcal{R}'$ the image of $\mathcal{R}$ under the map $\mathcal{V} \twoheadrightarrow \mathcal{W}$. We have a short exact sequence

$$0 \longrightarrow \mathcal{U} \cap \mathcal{R} \longrightarrow \mathcal{R} \longrightarrow \mathcal{R}' \longrightarrow 0,$$

Moreover, Proposition 2.2.14 shows that $\mathcal{U} \cap \mathcal{R}$ and $\mathcal{R}'$ are vector bundles on Z with

$$\mu(\mathcal{U} \cap \mathcal{R}) \leq \mu(\mathcal{U}) = \lambda \quad \text{and} \quad \mu(\mathcal{R}') \leq \mu(\mathcal{W}) = \lambda.$$

Hence we find $\mu(\mathcal{R}) \leq \lambda$ by Proposition 2.2.8 and in turn see that $\mathcal{V}$ is semistable of slope λ, thereby completing the proof. □

PROPOSITION 2.2.16. Given semistable vector bundles $\mathcal{V}$ and $\mathcal{W}$ on Z with $\mu(\mathcal{V}) > \mu(\mathcal{W})$, every $\mathcal{O}_Z$-module homomorphism from $\mathcal{V}$ to $\mathcal{W}$ is zero.

PROOF. Suppose for contradiction that we have a nonzero $\mathcal{O}_Z$-module map $f : \mathcal{V} \to \mathcal{W}$. Let $\mathcal{Q}$ denote the image of f. Proposition 2.2.14 shows that $\mathcal{Q}$ is a vector bundle on Z with

$$\mu(\mathcal{Q}) \leq \mu(\mathcal{W}) < \mu(\mathcal{V}). \tag{2.8}$$

Moreover, $\mathcal{Q}$ fits into a short exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow \mathcal{V} \xrightarrow{f} \mathcal{Q} \longrightarrow 0.$$

We have $\ker(f) \neq 0$ as $\mathcal{Q}$ and $\mathcal{V}$ are not isomorphic by the inequality (2.8). Hence we obtain the inequality $\mu(\ker(f)) \leq \mu(\mathcal{V})$ by the semistability of $\mathcal{V}$ and in turn find $\mu(\mathcal{Q}) \geq \mu(\mathcal{V})$ by Proposition 2.2.8, thereby deducing a desired contradiction by the inequality (2.8). □

Definition 2.2.17. Given a vector bundle $\mathcal{V}$ on Z, its *Harder-Narasimhan filtration* is a finite chain of vector bundles
$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}$$
which satisfies the following properties:

(i) Each $\mathcal{V}_i$ is a subbundle of $\mathcal{V}_{i+1}$.
(ii) The vector bundles $\mathcal{V}_1/\mathcal{V}_0, \cdots, \mathcal{V}_n/\mathcal{V}_{n-1}$ are semistable with
$$\mu(\mathcal{V}_1/\mathcal{V}_0) > \cdots > \mu(\mathcal{V}_n/\mathcal{V}_{n-1}).$$

Remark. It is not hard to see that each $\mathcal{V}_i$ must be a subbundle of $\mathcal{V}$.

Lemma 2.2.18. Every nonzero vector bundle $\mathcal{V}$ on Z admits a semistable subbundle $\mathcal{V}_1$ with $\mu(\mathcal{V}_1) \geq \mu(\mathcal{V})$ and $\mu(\mathcal{V}_1) > \mu(\mathcal{W})$ for each nonzero subbundle $\mathcal{W}$ of $\mathcal{V}/\mathcal{V}_1$.

Proof. Proposition 2.2.11 yields an integer $d_{\mathcal{V}}$ such that each nonzero subbundle $\mathcal{U}$ of $\mathcal{V}$ satisfies the inequalities $0 < \mathrm{rk}(\mathcal{U}) \leq \mathrm{rk}(\mathcal{V})$ and $\deg(\mathcal{U}) \leq d_{\mathcal{V}}$. Hence the set
$$S := \{\, q \in \mathbb{Q} : q = \mu(\mathcal{U}) \text{ for some nonzero subbundle } \mathcal{U} \text{ of } \mathcal{V} \,\}$$
is discrete and bounded above. In particular, S admits the maximum element λ. Let us take $\mathcal{V}_1$ to be an element of maximal rank in the set of subbundles of $\mathcal{V}$ with slope λ. The maximality of λ implies that $\mathcal{V}_1$ satisfies the inequality $\mu(\mathcal{V}_1) \geq \mu(\mathcal{V})$. Moreover, since every subbundle of $\mathcal{V}_1$ is a coherent $\mathcal{O}_Z$-module of $\mathcal{V}$, we use Proposition 2.2.9 and the maximality of λ to see that $\mathcal{V}_1$ is semistable. Let us now consider an arbitrary nonzero subbundle $\mathcal{W}$ of $\mathcal{V}/\mathcal{V}_1$ and denote by $\mathcal{W}'$ the preimage of $\mathcal{W}$ under the natural map $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}_1$. We observe that $\mathcal{W}'$ is a subbundle of $\mathcal{V}$ and also obtain a short exact sequence
$$0 \longrightarrow \mathcal{V}_1 \longrightarrow \mathcal{W}' \longrightarrow \mathcal{W} \longrightarrow 0.$$
In addition, we find $\mu(\mathcal{W}') < \lambda = \mu(\mathcal{V}_1)$ by the maximality of λ and $\mathcal{V}_1$. Hence we deduce the inequality $\mu(\mathcal{W}) < \mu(\mathcal{V}_1)$ from Proposition 2.2.8, thereby completing the proof. □

Remark. Our proof of Lemma 2.2.18 relies on the fact that the degree map takes values in the discrete group $\mathbb{Z}$. However, in the general context where the degree map takes values in an arbitrary totally ordered abelian group, we can still prove Lemma 2.2.18 and consequently show that all results from this subsection remain valid. We refer curious readers to the notes of Kedlaya [**Ked19**, Lemma 3.4.10 and Example 3.5.7] for details.

Lemma 2.2.19. Let $\mathcal{V}$ be a nonzero vector bundle on Z with a Harder-Narasimhan filtration
$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}.$$
For every semistable vector bundle $\mathcal{W}$ on Z with $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{W}, \mathcal{V}) \neq 0$, we have $\mu(\mathcal{W}) \leq \mu(\mathcal{V}_1)$.

Proof. Let us take a nonzero $\mathcal{O}_Z$-module map $f : \mathcal{W} \to \mathcal{V}$ and denote its image by $\mathcal{Q}$. Since $\mathcal{Q}$ is a nonzero coherent $\mathcal{O}_Z$-submodule of $\mathcal{V}$, we can find the smallest integer $i \geq 1$ with $\mathcal{Q} \subseteq \mathcal{V}_i$. We see that f induces a nonzero $\mathcal{O}_Z$-module map $\mathcal{W} \to \mathcal{V}_i \twoheadrightarrow \mathcal{V}_i/\mathcal{V}_{i-1}$ and in turn apply Proposition 2.2.16 to obtain the inequality
$$\mu(\mathcal{W}) \leq \mu(\mathcal{V}_i/\mathcal{V}_{i-1}) \leq \mu(\mathcal{V}_1),$$
thereby completing the proof. □

Remark. Lemma 2.2.19 does not hold without the semistability assumption on $\mathcal{W}$; for example, if we take $\mathcal{W} = \mathcal{V}_1 \oplus \mathcal{L}$ for some line bundle $\mathcal{L}$ on Z with $\mu(\mathcal{L}) > \mu(\mathcal{V}_1)$, we obtain a nonzero $\mathcal{O}_Z$-module map $\mathcal{W} \twoheadrightarrow \mathcal{V}_1 \hookrightarrow \mathcal{V}$ and also find $\mu(\mathcal{W}) > \mu(\mathcal{V}_1)$.

THEOREM 2.2.20 (Harder-Narasimhan [**HN75**]). Every vector bundle $\mathcal{V}$ on Z admits a unique Harder-Narasimhan filtration.

PROOF. If $\mathcal{V}$ is the zero bundle on Z, the assertion is trivial. Let us now assume the inequality $\mathrm{rk}(\mathcal{V}) > 0$ and proceed by induction on $\mathrm{rk}(\mathcal{V})$. Lemma 2.2.18 yields a semistable subbundle $\mathcal{V}_1$ of $\mathcal{V}$ with $\mu(\mathcal{V}_1) > \mu(\mathcal{U})$ for every nonzero subbundle $\mathcal{U}$ of $\mathcal{V}/\mathcal{V}_1$. By the induction hypothesis, the vector bundle $\mathcal{V}/\mathcal{V}_1$ on Z admits a unique Harder-Narasimhan filtration

$$0 = \mathcal{U}_1 \subset \cdots \subset \mathcal{U}_n = \mathcal{V}/\mathcal{V}_1. \tag{2.9}$$

For each $i \in \mathbb{Z}$ with $2 \leq i \leq n$, let us set $\mathcal{V}_i$ to be the preimage of $\mathcal{U}_i$ under the map $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}_1$. We see that each $\mathcal{V}_i/\mathcal{V}_{i-1}$ with $i \geq 2$ admits a natural isomorphism $\mathcal{V}_i/\mathcal{V}_{i-1} \cong \mathcal{U}_i/\mathcal{U}_{i-1}$. Moreover, we have $\mu(\mathcal{V}_1) > \mu(\mathcal{U}_2)$ whenever the Harder-Narasimhan filtration (2.9) is not trivial. Therefore $\mathcal{V}$ admits a Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}. \tag{2.10}$$

It remains to show that the chain (2.10) is a unique Harder-Narasimhan filtration of $\mathcal{V}$. Let us assume that $\mathcal{V}$ admits another Harder-Narasimhan filtration

$$0 = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_m = \mathcal{V}. \tag{2.11}$$

We note that $\mathcal{V}/\mathcal{W}_1$ admits a Harder-Narasimhan filtration

$$0 = \mathcal{W}_1/\mathcal{W}_1 \subset \cdots \mathcal{V}_m/\mathcal{W}_1 = \mathcal{V}/\mathcal{W}_1. \tag{2.12}$$

Since $\mathcal{V}_1$ and $\mathcal{W}_1$ are nonzero semistable subbundles of $\mathcal{V}$, Lemma 2.2.19 yields the inequalities $\mu(\mathcal{W}_1) \leq \mu(\mathcal{V}_1)$ and $\mu(\mathcal{V}_1) \leq \mu(\mathcal{W}_1)$. Hence we have

$$\mu(\mathcal{W}_1) = \mu(\mathcal{V}_1) > \mu(\mathcal{V}_2/\mathcal{V}_1) = \mu(\mathcal{U}_2/\mathcal{U}_1)$$

unless the Harder-Narasimhan filtration (2.9) is trivial. We deduce from Lemma 2.2.19 that the natural map $\mathcal{W}_1 \hookrightarrow \mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{V}_1$ is zero and in turn see that $\mathcal{V}_1$ contains $\mathcal{W}_1$. Similarly, we find that $\mathcal{W}_1$ contains $\mathcal{V}_1$. Now we obtain the identity $\mathcal{V}_1 = \mathcal{W}_1$, which in particular implies that the Harder-Narasimhan filtrations (2.9) and (2.12) must coincide. We see that each $\mathcal{W}_i$ with $i \geq 1$ is the preimage of $\mathcal{W}_i/\mathcal{W}_1 = \mathcal{U}_i$ under the natural surjection $\mathcal{V} \twoheadrightarrow \mathcal{V}/\mathcal{W}_1 = \mathcal{V}/\mathcal{V}_1$ and consequently deduce that the Harder-Narasimhan filtrations (2.10) and (2.11) coincide. □

Remark. A careful examination of our discussion shows that Theorem 2.2.20 is a formal consequence of Proposition 2.2.8 and Proposition 2.2.9. Hence we can extend Theorem 2.2.20 to every additive category which admits reasonable notions of rank and degree with appropriate analogues of Proposition 2.2.8 and Proposition 2.2.9. We refer curious readers to the notes of Kedlaya [**Ked19**, Definition 3.3.1] for a precise characterization of such a category, often referred to as a *slope category*.

Slope categories are prevalent in p-adic Hodge theory. We have already introduced two important examples, namely the category of filtered isocrystals over a p-adic field and the category of vector bundles on the Fargues-Fontaine curve. In the next section, we will explore the relationship between these categories to prove that every weakly admissible filtered isocrystal over a p-adic field is admissible.

It is worthwhile to mention that a slope category does not need to admit tensor products; indeed, our proof of Theorem 2.2.20 does not involve tensor products. Moreover, even for a slope category with tensor products, the Harder-Narasimhan filtrations may behave unfavorably under tensor products; for example, when Z is a projective curve over a field of positive characteristic, a result of Gieseker [**Gie73**] shows that the tensor product of semistable vector bundles on Z is not necessarily semistable. However, most slope categories in practice admit tensor products which exhibit nice properties in relation to the Harder-Narasimhan filtrations.

2.3. Classification of vector bundles

Our main goal for this subsection is to provide an explicit classification of vector bundles on the Fargues-Fontaine curve. For every integer $h > 0$, we write E_h for the degree h unramified extension of $\mathbb{Q}_p$.

Definition 2.3.1. Given an integer $h > 0$, the *degree h unramified cover* of X is the scheme

$$X_h := \mathrm{Proj}\,(P_h) \qquad \text{with} \qquad P_h := \bigoplus_{n \geq 0} B^{\varphi^h = p^n}.$$

LEMMA 2.3.2. Let m and n be integers with $m > 0$. Given a positive integer h and a nonzero homogeneous element $f \in P$, there exists a canonical isomorphism

$$B[1/f]^{\varphi^m = p^n} \otimes_{\mathbb{Q}_p} E_h \cong B[1/f]^{\varphi^{mh} = p^{nh}}.$$

PROOF. The group $\mathrm{Gal}(E_h/\mathbb{Q}_p)$ is cyclic of order h and admits a canonical generator γ induced by the p-th power map on $\mathbb{F}_{p^h}$. We see that $B[1/f]^{\varphi^{mh} = p^{nh}}$ is naturally a semilinear $\mathrm{Gal}(E_h/\mathbb{Q}_p)$-module with the action of γ given by $p^{-n}\varphi^m$. Hence we find

$$B[1/f]^{\varphi^m = p^n} = \left(B[1/f]^{\varphi^{mh} = p^{nh}}\right)^{\mathrm{Gal}(E_h/\mathbb{Q}_p)}$$

and in turn obtain the desired isomorphism by Lemma 2.4.16 in Chapter III. □

Remark. Proposition 1.3.13 and Lemma 2.3.2 together imply that $B^{\varphi^h = 1}$ is canonically isomorphic to E_h.

PROPOSITION 2.3.3. For every integer $h > 0$, there exists a natural isomorphism

$$X_h \cong X \times_{\mathrm{Spec}\,(\mathbb{Q}_p)} \mathrm{Spec}\,(E_h).$$

PROOF. Since Lemma 2.3.2 yields a canonical isomorphism

$$B^{\varphi = p^n} \otimes_{\mathbb{Q}_p} E_h \cong B^{\varphi^h = p^{nh}} \qquad \text{for every } n \in \mathbb{Z},$$

we obtain a natural isomorphism

$$X_h = \mathrm{Proj}\,\left(\bigoplus_{n \geq 0} B^{\varphi^h = p^n}\right) \cong \mathrm{Proj}\,\left(\bigoplus_{n \geq 0} B^{\varphi^h = p^{nh}}\right) \cong \mathrm{Proj}\,\left(P \otimes_{\mathbb{Q}_p} E_h\right)$$

and consequently establish the desired assertion. □

LEMMA 2.3.4. Given an integer $h > 0$, the scheme $X_h = \mathrm{Proj}\,(P_h)$ admits an affine open cover given by the standard open subschemes associated to homogeneous elements in P.

PROOF. Take an arbitrary point x on X_h. We wish to show that x lies in a standard open subscheme of $X_h = \mathrm{Proj}\,(P_h)$. Proposition 2.3.3 yields a natural morphism

$$\pi_h : X_h \cong X \times_{\mathrm{Spec}\,(\mathbb{Q}_p)} \mathrm{Spec}\,(E_h) \longrightarrow X.$$

Let us take a nonzero homogeneous element $f \in P$ such that $\pi_h(x)$ lies in the standard open subscheme $U(f) := \mathrm{Spec}\,(B[1/f]^{\varphi=1})$ of $X = \mathrm{Proj}\,(P)$. We apply Lemma 2.3.2 to obtain a canonical isomorphism

$$B[1/f]^{\varphi=1} \otimes_{\mathbb{Q}_p} E_h \cong B[1/f]^{\varphi^h=1}$$

and in turn identify $\pi_h^{-1}(U(f))$ with the standard open subscheme $U_h(f) := \mathrm{Spec}\,(B[1/f]^{\varphi^h=1})$ of $X_h = \mathrm{Proj}\,(P_h)$. Now the desired assertion is evident as x lies in $\pi_h^{-1}(U(f))$. □

Remark. In fact, by Proposition 1.4.19 we can take f to be an element in $B^{\varphi=p}$.

We state the following generalization of Proposition 1.4.19 without a proof.

PROPOSITION 2.3.5. Given integers $h, n \geq 1$, every nonzero $f \in B^{\varphi^h = p^n}$ admits an expression

$$f = f_1 \cdots f_n \quad \text{ with } f_i \in B^{\varphi^h = p}$$

where the factors are unique up to $E_h^\times$-multiple.

Remark. Let us briefly sketch the proof of Proposition 2.3.5. The theory of Lubin-Tate formal groups yields a unique 1-dimensional p-divisible formal group law μ_{LT} over $\mathcal{O}_{E_h}$ with $[p]_{\mu_{\mathrm{LT}}}(t) = pt + t^{p^h}$. Denote by G_{LT} the associated p-divisible group over $\mathcal{O}_{E_h}$. By means of the logarithm for G_{LT}, we can construct a group homomorphism

$$\log_h : G_{\mathrm{LT}}(\mathcal{O}_F) := \varprojlim_i G_{\mathrm{LT}}(\mathcal{O}_F/\mathfrak{m}_F^i \mathcal{O}_F) \longrightarrow B^{\varphi^h = p}.$$

It turns out that all results from §1.4 remain valid with $\mathbb{Q}_p$, log, $1 + \mathfrak{m}_F^*$, φ, ϕ, P, and X respectively replaced by E_h, $\log_h$, $G_{\mathrm{LT}}(\mathcal{O}_F)$, φ^h, ϕ^h, P_h, and X_h. We refer readers to the article of Fargues-Fontaine [**FF18**, §6.2] or the notes of Lurie [**Lur**, Lectures 22-26] for details.

Definition 2.3.6. Given integers d and h with $h > 0$, the *d-fold Serre twist* of $\mathcal{O}_{X_h}$ is the quasicoherent $\mathcal{O}_{X_h}$-module $\mathcal{O}_h(d) = \mathcal{O}_{X_h}(d)$ associated to $P_h(d) := \bigoplus_{n \geq 0} B^{\varphi^h = p^{d+n}}$.

LEMMA 2.3.7. Given integers d and h with $h > 0$, the $\mathcal{O}_{X_h}$-module $\mathcal{O}_h(d)$ is a line bundle on X_h with a canonical isomorphism $\mathcal{O}_h(d) \cong \mathcal{O}_h(1)^{\otimes d}$.

PROOF. The assertion follows from Proposition 2.3.5 and a standard fact stated in the Stacks project [**Sta**, Tag 01MT]. □

Definition 2.3.8. Let h be a positive integer.

(1) Given an integer $r > 0$, we refer to the morphism $\pi_{rh,h} : X_{rh} \to X_h$ induced by the natural embedding $P_h \hookrightarrow P_{rh}$ as the *standard projection* from X_{rh} to X_h.
(2) Given integers d, r with $r > 0$, we refer to $\mathcal{O}_h(d, r) := (\pi_{rh,h})_* \mathcal{O}_{rh}(d)$ as the *standard $\mathcal{O}_{X_h}$-module* of type (d, r).

PROPOSITION 2.3.9. Given integers d, h, and r with $h, r > 0$, the $\mathcal{O}_{X_h}$-module $\mathcal{O}_h(d, r)$ is a vector bundle on X_h of rank r.

PROOF. Proposition 2.3.3 shows that the morphism $\pi_{rh,h}$ is finite of degree r. Hence the desired assertion follows from Lemma 2.3.7. □

PROPOSITION 2.3.10. Given integers d, h, and r with $h, r > 0$, we have a natural isomorphism

$$(\pi_{hn,h})^* \mathcal{O}_h(d, r) \cong \mathcal{O}_{hn}(dn, r) \quad \text{ for every } n > 0.$$

PROOF. Let us take an arbitrary nonzero homogeneous element $f \in P$. We write $U_h(f)$ and $U_{hn}(f)$ respectively for the standard open subschemes of X_h and X_{hn} associated to f. We apply Lemma 2.3.2 and Proposition 2.3.3 to find

$$\begin{aligned}
(\pi_{hn,h})^* \mathcal{O}_h(d, r) \left(U_{hn}(f)\right) &\cong \mathcal{O}_h(d, r) \left(U_h(f)\right) \otimes_{B[1/f]^{\varphi^h = 1}} B[1/f]^{\varphi^{hn} = 1} \\
&\cong B[1/f]^{\varphi^{hr} = p^d} \otimes_{B[1/f]^{\varphi^h = 1}} \left(B[1/f]^{\varphi^h = 1} \otimes_{\mathbb{Q}_p} E_n\right) \\
&\cong B[1/f]^{\varphi^{hnr} = p^{dn}} \\
&\cong \mathcal{O}_{hn}(dn, r) \left(U_{hn}(f)\right).
\end{aligned}$$

Hence we establish the desired assertion by Lemma 2.3.4. □

PROPOSITION 2.3.11. Given integers d, h, and r with $h, r > 0$, we have a natural isomorphism

$$\mathcal{O}_h(dn, dn) \cong \mathcal{O}_h(d, r)^{\oplus n} \quad \text{for every } n > 0.$$

PROOF. Proposition 2.3.10 yields a natural isomorphism

$$\mathcal{O}_h(dn, rn) = (\pi_{hr,h})_*(\pi_{hnr,hr})_*\mathcal{O}_{hnr}(dn) \cong (\pi_{hr,h})_*(\pi_{hnr,hr})_*(\pi_{hnr,hr})^*\mathcal{O}_{hr}(d).$$

Moreover, we use the projection formula and Proposition 2.3.3 to find

$$\begin{aligned}(\pi_{hnr,hr})_*(\pi_{hnr,hr})^*\mathcal{O}_{hr}(d) &\cong (\pi_{hnr,hr})_*\mathcal{O}_{X_{hnr}} \otimes_{\mathcal{O}_{X_{hr}}} \mathcal{O}_{hr}(d) \\ &\cong \mathcal{O}_{X_{hr}}^{\oplus n} \otimes_{\mathcal{O}_{X_{hr}}} \mathcal{O}_{hr}(d) \cong \mathcal{O}_{hr}(d)^{\oplus n}.\end{aligned}$$

Now the desired assertion is evident. □

PROPOSITION 2.3.12. Given an integer $h > 0$, there exists a canonical isomorphism

$$\mathcal{O}_h(d_1, r_1) \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(d_2, r_2) \cong \mathcal{O}_h(d_1r_2 + d_2r_1, r_1r_2)$$

for every $d_1, d_2, r_1, r_2 \in \mathbb{Z}$ with $r_1, r_2 > 0$.

PROOF. Let g and l respectively denote the greatest common divisor and the least common multiple of r_1 and r_2. Since $r_1' := r_1/g$ and $r_2' := r_2/g$ are relatively prime integers, the field extensions E_{r_1h} and E_{r_2h} of E_h yield a natural isomorphism $E_{lh} \cong E_{r_1h} \otimes_{E_{gh}} E_{r_2h}$. Hence by Proposition 2.3.3, we obtain a cartesian diagram

$$\begin{array}{ccc} X_{lh} & \xrightarrow{\pi_{lh,r_2h}} & X_{r_2h} \\ {\scriptstyle \pi_{lh,r_1h}}\downarrow & & \downarrow{\scriptstyle \pi_{r_2h,gh}} \\ X_{r_1h} & \xrightarrow{\pi_{r_1h,gh}} & X_{gh} \end{array}$$

where all arrows are finite étale. Moreover, we apply the Künneth formula, Lemma 2.3.7, and Proposition 2.3.10 to obtain an identification

$$\begin{aligned}\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} \mathcal{O}_{gh}(d_2, r_2') &= (\pi_{r_1h,gh})_*(\mathcal{O}_{r_1h}(d_1)) \otimes_{\mathcal{O}_{X_{gh}}} (\pi_{r_2h,gh})_*(\mathcal{O}_{r_2h}(d_2)) \\ &\cong (\pi_{lh,gh})_*\left((\pi_{lh,r_1h})^*\mathcal{O}_{r_1h}(d_1) \otimes_{\mathcal{O}_{X_{lh}}} (\pi_{lh,r_2h})^*\mathcal{O}_{r_2h}(d_2)\right) \\ &\cong (\pi_{lh,gh})_*\left(\mathcal{O}_{lh}(d_1r_1') \otimes_{\mathcal{O}_{X_{lh}}} \mathcal{O}_{lh}(d_2r_2')\right) \\ &\cong (\pi_{lh,gh})_*\mathcal{O}_{lh}(d_1r_1' + d_2r_2') \\ &= \mathcal{O}_{gh}(d_1r_1' + d_2r_2', r_1'r_2').\end{aligned}$$

Now we use the projection formula, Proposition 2.3.10, and Proposition 2.3.11 to find

$$\begin{aligned}\mathcal{O}_h(d_1, r_1) \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(d_2, r_2) &= (\pi_{gh,h})_*\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(d_2, r_2) \\ &\cong (\pi_{gh,h})_*\left(\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} (\pi_{gh,h})^*\mathcal{O}_h(d_2, r_2)\right) \\ &\cong (\pi_{gh,h})_*\left(\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} \mathcal{O}_{gh}(d_2g, r_2)\right) \\ &\cong (\pi_{gh,h})_*\left(\mathcal{O}_{gh}(d_1, r_1') \otimes_{\mathcal{O}_{X_{gh}}} \mathcal{O}_{gh}(d_2, r_2')^{\oplus g}\right) \\ &\cong (\pi_{gh,h})_*\mathcal{O}_{gh}(d_1r_1' + d_2r_2', r_1'r_2')^{\oplus g} \\ &= \mathcal{O}_h(d_1r_1' + d_2r_2', gr_1'r_2')^{\oplus g} \\ &\cong \mathcal{O}_h(d_1r_1 + d_2r_2, r_1r_2),\end{aligned}$$

thereby completing the proof. □

PROPOSITION 2.3.13. Given integers d, h, and r with $h, r > 0$, the dual $\mathcal{O}_h(d,r)^\vee$ of $\mathcal{O}_h(d,r)$ admits a canonical isomorphism

$$\mathcal{O}_h(d,r)^\vee \cong \mathcal{O}_h(-d,r).$$

PROOF. Proposition 2.3.11 and Proposition 2.3.12 together yield a natural isomorphism

$$\mathcal{O}_h(d,r) \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(-d,r) \cong \mathcal{O}_{X_h}^{\oplus r^2}.$$

In addition, since $\mathcal{O}_{X_h}$ is isomorphic to its dual, there exists a canonical perfect pairing

$$\mathcal{O}_{X_h}^{\oplus r} \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_{X_h}^{\oplus r} \longrightarrow \mathcal{O}_{X_h}.$$

Hence we use the natural isomorphism $\mathcal{O}_{X_h}^{\oplus r^2} \cong \mathcal{O}_{X_h}^{\oplus r} \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_{X_h}^{\oplus r}$ to obtain a perfect pairing

$$\mathcal{O}_h(d,r) \otimes_{\mathcal{O}_{X_h}} \mathcal{O}_h(-d,r) \longrightarrow \mathcal{O}_{X_h},$$

thereby establishing the desired assertion. □

PROPOSITION 2.3.14. Let d and r be integers with $r > 0$.

(1) The vector bundle $\mathcal{O}(d,r) := \mathcal{O}_1(d,r)$ on X is semistable of rank r and degree d.
(2) If d and r are relatively prime, the vector bundle $\mathcal{O}(d,r)$ is stable.

PROOF. Proposition 2.3.11 and Proposition 2.3.12 together yield a natural isomorphism

$$\mathcal{O}(d,r)^{\otimes r} \cong \mathcal{O}(dr^r, r^r) \cong \mathcal{O}(d)^{\oplus r^r}.$$

In addition, we find $\deg\left(\mathcal{O}(d)^{\oplus r^r}\right) = dr^r$ by Proposition 2.2.8. Therefore we deduce from Proposition 2.2.7 and Proposition 2.3.9 that $\mathcal{O}(d,r)$ has rank r and degree d.

Let us now consider an arbitrary nonzero subbundle $\mathcal{V}$ of $\mathcal{O}(d,r)$ with $\mathcal{V} \neq \mathcal{O}(d,r)$. We may regard $\mathcal{V}^{\otimes r}$ as a coherent $\mathcal{O}_X$-submodule of $\mathcal{O}(d,r)^{\otimes r}$. Moreover, Example 2.2.13 and Proposition 2.2.15 together show that $\mathcal{O}(d,r)^{\otimes r} \cong \mathcal{O}(d)^{\oplus r^r}$ is semistable. Hence we apply Proposition 2.2.7 and Proposition 2.2.14 to find

$$\mu(\mathcal{V}) = \mu(\mathcal{V}^{\otimes r})/r \leq \mu(\mathcal{O}(d,r)^{\otimes r})/r = \mu(\mathcal{O}(d,r)),$$

thereby deducing that $\mathcal{O}(d,r)$ is semistable. If d and r are relatively prime, we also find $\mu(\mathcal{V}) \neq d/r$ by the inequality $\mathrm{rk}(\mathcal{V}) < \mathrm{rk}(\mathcal{O}(d,r)) = r$ and in turn see that $\mathcal{O}(d,r)$ is stable. □

Remark. Proposition 2.3.14 remains valid if we replace $\mathcal{O}(d,r)$ and X respectively with $\mathcal{O}_h(d,r)$ and X_h for an arbitrary integer $h > 0$, as X_h turns out to be a complete abstract curve.

Definition 2.3.15. Given a rational number $\lambda = d/r$ written in a reduced form with $r > 0$, we refer to $\mathcal{O}(\lambda) := \mathcal{O}_1(d,r)$ as the *canonical stable bundle of slope* λ on X.

PROPOSITION 2.3.16. Let λ be a rational number.

(1) The dual $\mathcal{O}(\lambda)^\vee$ of $\mathcal{O}(\lambda)$ admits a canonical isomorphism $\mathcal{O}(\lambda)^\vee \cong \mathcal{O}(-\lambda)$.
(2) Given a rational number λ', there exists a natural isomorphism

$$\mathcal{O}(\lambda) \otimes_{\mathcal{O}_X} \mathcal{O}(\lambda') \cong \mathcal{O}(\lambda + \lambda')^{\oplus n} \quad \text{for some } n \geq 1.$$

PROOF. Statement (1) is a special case of Proposition 2.3.13. Statement (2) follows from Proposition 2.3.11 and Proposition 2.3.12. □

Remark. In Statement (2), if we write $\lambda = d/r$ and $\lambda' = d'/r'$ in reduced form, n is equal to the greatest common divisor of $dr' + d'r$ and rr'.

In order to explain the classification theorem for vector bundles on the Fargues-Fonaine curve, we invoke the following crucial result without a proof.

PROPOSITION 2.3.17. Let λ be a rational number.

(1) A vector bundle on X is semistable of slope λ if and only if it is isomorphic to $\mathcal{O}(\lambda)^{\oplus n}$ for some $n \geq 1$.
(2) If we have $\lambda \geq 0$, the cohomology group $H^1(X, \mathcal{O}(\lambda))$ vanishes.

Remark. Statement (2) is relatively easy to prove. Let us write $\lambda = d/r$ in reduced form. We can adjust our argument in §2.1 to show that Theorem 2.1.8 is valid with $\mathcal{O}_r(d)$ and X_r respectively in place of $\mathcal{O}(d)$ and X. Hence if λ is nonnegative, we find

$$H^1(X, \mathcal{O}(\lambda)) = H^1(X, (\pi_r)_*\mathcal{O}_r(d)) \cong H^1(X_r, \mathcal{O}_r(d)) = 0.$$

On the other hand, statement (1) is one of the most technical results from the original work of Fargues-Fontaine [**FF18**]. Its proof employs a series of dévissage arguments to deduce the assertion from some deep results about Lubin-Tate p-disivible groups due to Drinfeld [**Dri76**], Laffaille [**Laf85**], and Gross-Hopkins [**GH94**]. We refer curious readers to the survey article of Fargues-Fontaine [**FF14**, §6] for an excellent exposition of the proof.

THEOREM 2.3.18 (Fargues-Fontaine [**FF18**]). Every vector bundle $\mathcal{V}$ on X admits a direct sum decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{n} \mathcal{O}(\lambda_i)^{\oplus m_i} \quad \text{with } \lambda_i \in \mathbb{Q}.$$

PROOF. Theorem 2.2.20 shows that $\mathcal{V}$ admits a unique Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V}.$$

We wish to show that the Harder-Narasimhan filtration splits. If we have $n = 0$, the assertion is trivial. Let us henceforth assume the inequality $n > 0$ and proceed by induction on n. Proposition 2.3.17 implies that each $\mathcal{V}_i/\mathcal{V}_{i-1}$ admits an isomorphism

$$\mathcal{V}_i/\mathcal{V}_{i-1} \simeq \mathcal{O}(\lambda_i)^{\oplus m_i} \quad \text{with } \lambda_i \in \mathbb{Q}.$$

In addition, by the induction hypothesis, the Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_{n-1}$$

splits into a direct sum decomposition

$$\mathcal{V}_{n-1} \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}(\lambda_i)^{\oplus m_i}.$$

Meanwhile, for each $i = 1, \cdots, n$, we apply Proposition 2.3.16 to find

$$\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}(\lambda_n), \mathcal{O}(\lambda_i)) \cong H^1(X, \mathcal{O}(\lambda_i) \otimes_{\mathcal{O}_X} \mathcal{O}(\lambda_n)^\vee) \cong H^1(X, \mathcal{O}(\lambda_i - \lambda_n)^{\oplus n_i}) \quad \text{with } n_i > 0$$

and in turn use Proposition 2.3.17 to see that $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{O}(\lambda_n), \mathcal{O}(\lambda_i))$ vanishes. Hence we deduce that $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{V}/\mathcal{V}_{n-1}, \mathcal{V}_{n-1})$ also vanishes, thereby establishing the desired assertion. □

Remark. Theorem 2.3.18 is an analogue of the fact that every vector bundle $\mathcal{V}$ on $\mathbb{P}^1_{\mathbb{C}}$ admits a direct sum decomposition

$$\mathcal{V} \simeq \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(d_i)^{\oplus m_i} \quad \text{with } d_i \in \mathbb{Z}$$

as proved by Grothendieck [**Gro57**].

3. Applications to p-adic representations

In this section, we establish some fundamental results about p-adic representations and period rings by means of the Fargues-Fontaine curve and related objects. The primary references for this section are the survey articles of Fargues-Fontaine [**FF12**] and Morrow [**Mor19**].

3.1. Geometrization of p-adic period rings

Throughout this section, we let K be a p-adic field with absolute Galois group Γ_K and residue field k. In addition, we write K_0 for the fraction field of $W(k)$.

Proposition 3.1.1. The tilt of $\mathbb{C}_K$ is algebraically closed.

Proof. Let $f(z)$ be an arbitrary monic polynomial of degree $d > 0$ over $\mathbb{C}_K^\flat$. We wish to show that $f(z)$ has a root in $\mathbb{C}_K^\flat$. Take a nonzero element a in the maximal ideal of $\mathcal{O}_{\mathbb{C}_K^\flat}$. We may replace $f(z)$ by $a^{md}f(z/a^m)$ for some sufficiently large $m \in \mathbb{Z}$ to assume that $f(z)$ is a polynomial over $\mathcal{O}_{\mathbb{C}_K^\flat}$. Let us write

$$f(z) = z^d + c_1 z^{d-1} + \cdots + c_d \quad \text{with } c_i \in \mathcal{O}_{\mathbb{C}_K^\flat}.$$

Proposition 2.1.7 in Chapter III yields a natural isomorphism

$$\mathcal{O}_{\mathbb{C}_K^\flat} \cong \varprojlim_{x \mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}, \tag{3.1}$$

via which we identify each c_i with a sequence $(c_{i,n})_{n\geq 0}$ in $\mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. Choose a lift $\widetilde{c_{i,n}} \in \mathcal{O}_{\mathbb{C}_K}$ of each $c_{i,n}$. In addition, for each $n \geq 0$ we set

$$f_n(z) := z^d + c_{1,n}z^{d-1} + \cdots + c_{d,n} \quad \text{and} \quad \widetilde{f_n}(z) := z^d + \widetilde{c_{1,n}}z^{d-1} + \cdots + \widetilde{c_{d,n}}.$$

For each $n \geq 1$, we have

$$f_{n-1}(z^p) = z^{dp} + c_{1,n}^p z^{(d-1)p} + \cdots + c_{d,n}^p = \left(z^d + c_{1,n}z^{d-1} + \cdots + c_{d,n}\right)^p = f_n(z)^p. \tag{3.2}$$

Meanwhile, since $\mathbb{C}_K$ is algebraically closed by Proposition 3.1.10 in Chapter II, each $\widetilde{f_n}(z)$ admits a factorization

$$\widetilde{f_n}(z) = (z - \alpha_{n,1}) \cdots (z - \alpha_{n,d}) \quad \text{with } \alpha_{n,j} \in \mathcal{O}_{\mathbb{C}_K}.$$

Let us denote by $\overline{\alpha_{n,j}}$ the image of each $\alpha_{n,j}$ under the natural surjection $\mathcal{O}_{\mathbb{C}_K} \twoheadrightarrow \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}$. The identity (3.2) shows that each $\overline{\alpha_{n,j}}$ with $n \geq 1$ satisfies the equality

$$f_{n-1}(\overline{\alpha_{n,j}}^p) = f_n(\overline{\alpha_{n,j}})^p = 0$$

and in turn yields the relation

$$\widetilde{f_{n-1}}(\alpha_{n,j}^p) = (\alpha_{n,j}^p - \alpha_{n-1,1}) \cdots (\alpha_{n,j}^p - \alpha_{n-1,d}) \in p\mathcal{O}_{\mathbb{C}_K}.$$

Now for each $\alpha_{n,j}$ with $n \geq 1$, we find $m \in \mathbb{Z}$ with $\alpha_{n,j}^p - \alpha_{n-1,m} \in p^{1/d}\mathcal{O}_{\mathbb{C}_K}$ and thus obtain the identity $\overline{\alpha_{n,j}}^{p^d} = \overline{\alpha_{n-1,m}}^{p^{d-1}}$ by Proposition 2.1.6 in Chapter III. We deduce that there exists a sequence of integers (j_n) with $\overline{\alpha_{n,j_n}}^{p^d} = \overline{\alpha_{n-1,j_{n-1}}}^{p^{d-1}}$ for every $n \geq 1$. Let us take the element $\overline{\alpha} \in \mathcal{O}_{\mathbb{C}_K^\flat}$ given by the sequence $\left(\overline{\alpha_{n+d-1,j_{n+d-1}}}^{p^{d-1}}\right)_{n\geq 0}$ via the isomorphism (3.1). We apply the identity (3.2) to find

$$f_n\left(\overline{\alpha_{n+d-1,j_{n+d-1}}}^{p^{d-1}}\right) = f_{n+d-1}\left(\overline{\alpha_{n+d-1,j_{n+d-1}}}\right) = 0 \quad \text{for every } n \geq 0$$

and in turn see that $\overline{\alpha}$ is a root of f, thereby completing the proof. □

Remark. Proposition 3.1.1 is a special case of the tilting equivalence for perfectoid fields.

For the rest of this section, we take $F = \mathbb{C}_K^\flat$ and regard $\mathbb{C}_K$ as an untilt of F. In addition, we fix a distinguished element $\xi = [p^\flat] - p \in A_{\mathrm{inf}}$ for some $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$.

PROPOSITION 3.1.2. There exists a refinement of the discrete valuation topology on B_{dR}^+ with the following properties:

(i) The subring A_{inf} of B_{dR}^+ is closed.
(ii) The map $\theta_{\mathbb{C}_K}[1/p] : A_{\mathrm{inf}}[1/p] \twoheadrightarrow \mathbb{C}_K$ is continuous and open with respect to the p-adic topology on $\mathbb{C}_K$.
(iii) The tilted logarithm yields a continuous map $\log : \mathbb{Z}_p(1) \to B_{\mathrm{dR}}^+$ via natural injective maps $\mathbb{Z}_p(1) \hookrightarrow 1 + \mathfrak{m}_F$ and $B^{\varphi=p} \hookrightarrow B_{\mathrm{dR}}^+$.
(iv) The ring B_{dR}^+ is complete.

PROOF. Proposition 1.3.20 shows that B_{dR}^+ is a discrete valuation ring with uniformizer ξ. Proposition 1.3.20 also yields a natural isomorphism

$$B_{\mathrm{dR}}^+ \cong \varprojlim_i B/\ker(\widehat{\theta_{\mathbb{C}_K}})^i,$$

via which we endow B_{dR}^+ with the topology induced by B. This topology refines the discrete valuation topology as ξ generates $\ker(\widehat{\theta_{\mathbb{C}_K}})$ by Proposition 1.3.17.

We observe by Proposition 1.3.10 that A_{inf} is closed in B and thus obtain property (i). In addition, we establish property (ii) as an immediate consequence of Proposition 1.2.16. Meanwhile, since Proposition 1.4.3 yields a continuous $\mathbb{Z}_p$-linear injective map $\mathbb{Z}_p(1) \hookrightarrow 1+\mathfrak{m}_F$, we deduce property (iii) from Proposition 1.3.25 and Proposition 1.4.17. Therefore it remains to verify property (iv).

Proposition 1.2.16 implies that $\xi B = \ker(\widehat{\theta_{\mathbb{C}_K}})$ is closed in B. We deduce that each $\xi^i B = \ker(\widehat{\theta_{\mathbb{C}_K}})^i$ is closed in B and in turn find that each $B/\ker(\widehat{\theta_{\mathbb{C}_K}})^i$ is complete. Hence we conclude that B_{dR}^+ is complete as desired. □

Remark. Proposition 3.1.2 establishes Proposition 2.2.20 in Chapter III. Our proof does not rely on any results which we stated without a proof.

PROPOSITION 3.1.3. There exists a unique closed point ∞ on X whose associated prime ideal in P contains every cyclotomic uniformizer of B_{dR}^+ as a generator.

PROOF. Proposition 2.2.25 in Chapter III shows that a cyclotomic uniformizer of B_{dR}^+ is unique up to $\mathbb{Z}_p^\times$-multiple. In addition, Proposition 3.1.2 implies that every cyclotomic uniformizer of B_{dR}^+ is an element in $B^{\varphi=p}$. Therefore we establish the desired assertion by Proposition 1.4.17 and Theorem 1.4.22. □

Definition 3.1.4. We refer to the closed point ∞ on X given by Proposition 3.1.3 as the *distinguished point* on X and write $U := X \setminus \{\infty\}$.

PROPOSITION 3.1.5. The completed local ring at ∞ is naturally isomorphic to B_{dR}^+.

PROOF. Given a $\mathbb{Z}_p$-basis element $\varepsilon \in \mathbb{Z}_p(1)$, we find $\widehat{\theta_{\mathbb{C}_K}}(\log(\varepsilon)) = 0$ by Proposition 1.4.3 and in turn deduce from Proposition 1.4.13 that $\mathbb{C}_K$ represents $\phi^n(y_\varepsilon) \in Y$ for some $n \in \mathbb{Z}$. Hence the assertion follows from Theorem 1.4.22. □

Remark. Proposition 1.3.20 and Proposition 3.1.5 together show that the residue field of ∞ is naturally isomorphic to $\mathbb{C}_K$.

Definition 3.1.6. Given a real number ρ with $0 < \rho < 1$, we refer to the closure of $A_{\text{inf}}[1/p]$ in $B_{[\rho,\rho]}$ to be the *ring of Gauss normally integral elements in* $B_{[\rho,\rho]}$, denoted by B^+_ρ.

Lemma 3.1.7. Given a closed interval $[a, b] \subseteq (0, 1)$, there exists a real number $\lambda > 0$ with

$$|f|_b \leq |f|_a^\lambda \quad \text{for every } f \in A_{\text{inf}}[1/p].$$

Proof. Let us take the positive real number $\lambda \leq 1$ with $a^\lambda = b$. For every $f \in A_{\text{inf}}[1/p]$ with a Teichmüller expansion $f = \sum [c_n]p^n$, we find

$$|f|_b = \sup_{n\in\mathbb{Z}}(|c_n|\, b^n) = \sup_{n\in\mathbb{Z}}\left(|c_n|\, a^{\lambda n}\right) \leq \sup_{n\in\mathbb{Z}}\left(|c_n|^\lambda\, a^{\lambda n}\right) = \sup_{n\in\mathbb{Z}}\left((|c_n|\, a^n)^\lambda\right) = |f|_a^\lambda$$

as desired. □

Proposition 3.1.8. Given a closed interval $[a, b] \subseteq (0, 1)$, there exists a canonical continuous embedding $B^+_a \hookrightarrow B^+_b$.

Proof. By Lemma 3.1.7, every Cauchy sequence in $A_{\text{inf}}[1/p]$ under the Gauss a-norm is Cauchy under the Gauss b-norm. Therefore we obtain a canonical continuous ring homomorphism $B^+_a \to B^+_b$. It remains to show that this map is injective. Take an arbitrary nonzero element $f \in B^+_a$ and denote by f' its image in B^+_b. Lemma 3.1.7 implies that f is naturally an element in $B_{[a,b]}$. Since $\mathcal{L}_f^{[a,b]}$ takes finite values by Proposition 1.3.6, we find $|f'|_b = |f|_b \neq 0$ and in turn deduce that f' is nonzero, thereby completing the proof. □

Lemma 3.1.9. The completion of a normed $\mathbb{Q}_p$-space V is naturally isomorphic to $\widehat{V_0}[1/p]$, where $\widehat{V_0}$ denotes the p-adic completion of the closed unit disk V_0 in V.

Proof. Since p is topologically nilpotent in $\mathbb{Q}_p$, a sequence (v_n) in V is Cauchy under the norms if and only if $(p^{-m}v_n)$ is p-adically Cauchy in V_0 for some $m \geq 0$. □

Proposition 3.1.10. For every $c \in \mathcal{O}_F^\times$, there exists a canonical topological isomorphism

$$B^+_{|c|} \cong \widehat{A_{\text{inf}}[[c]/p]}[1/p]$$

where $\widehat{A_{\text{inf}}[[c]/p]}$ denotes the p-adic completion of $A_{\text{inf}}[[c]/p]$.

Proof. The ring $B^+_{|c|}$ is naturally isomorphic to the completion of $A_{\text{inf}}[1/p]$ under the Gauss $|c|$-norm. In light of Lemma 3.1.9, it suffices to establish the identification

$$A_{\text{inf}}[[c]/p] = \left\{ f \in A_{\text{inf}}[1/p] : |f|_{|c|} \leq 1 \right\}.$$

Since we have $|[c]/p|_{|c|} = 1$, every $f \in A_{\text{inf}}[[c]/p]$ satisfies the inequality $|f|_{|c|} \leq 1$. Hence it remains to show that every $f \in A_{\text{inf}}[1/p]$ with $|f|_{|c|} \leq 1$ lies in $A_{\text{inf}}[[c]/p]$. Let us write

$$f = \sum_{n<0}[c_n]p^n + \sum_{n\geq 0}[c_n]p^n \quad \text{with } c_n \in \mathcal{O}_F. \tag{3.3}$$

For every $n \in \mathbb{Z}$, we have $|c_n|\,|c|^n \leq |f|_{|c|} \leq 1$ or equivalently $c_n c \in \mathcal{O}_F$. Hence we find

$$[c_n]p^n = [c_n c^n] \cdot ([c]/p)^{-n} \in A_{\text{inf}}[[c]/p] \quad \text{for every } n < 0$$

and in turn deduce that the first sum in the identity (3.3) yields an element in $A_{\text{inf}}[[c]/p]$ for having finitely many nonzero terms. Now we obtain the desired assertion by observing that the second sum in the identity (3.3) yields an element in A_{inf}. □

PROPOSITION 3.1.11. There exist natural continuous injective maps

$$B^+_{1/p^p} \hookrightarrow B^+_{\text{cris}} \quad \text{and} \quad B^+_{\text{cris}} \hookrightarrow B^+_{1/p}.$$

PROOF. Proposition 3.1.10 yields natural isomorphisms

$$B^+_{1/p^p} \cong \widehat{A_{\text{inf}}[[(p^\flat)^p]/p]}[1/p] \quad \text{and} \quad B^+_{1/p} \cong \widehat{A_{\text{inf}}[[p^\flat]/p]}[1/p],$$

where $\widehat{A_{\text{inf}}[[(p^\flat)^p]/p]}$ and $\widehat{A_{\text{inf}}[[p^\flat]/p]}$ respectively denote the p-adic completions of $A_{\text{inf}}[[(p^\flat)^p]/p]$ and $A_{\text{inf}}[[p^\flat]/p]$. Meanwhile, we have the identification $B^+_{\text{cris}} = A_{\text{cris}}[1/p]$ where A_{cris} is the p-adic completion of A^0_{cris}. Hence it suffices to establish the relation

$$A_{\text{inf}}[[(p^\flat)^p]/p] \subseteq A^0_{\text{cris}} \subseteq A_{\text{inf}}[[p^\flat]/p].$$

We see that A^0_{cris} contains $A_{\text{inf}}[[(p^\flat)^p]/p]$ as we have

$$\frac{[p^\flat]^p}{p} = \frac{(\xi+p)^p}{p} = (p-1)! \cdot \frac{\xi^p}{p!} + \sum_{i=1}^{p} \binom{p}{i} p^{i-1}\xi^{p-i} \in A^0_{\text{cris}}.$$

In addition, we find

$$\frac{\xi^n}{n!} = \frac{([p^\flat]-p)^n}{n!} = \frac{p^n}{n!}\left(\frac{[p^\flat]}{p} - 1\right)^n \in A_{\text{inf}}[[p^\flat]/p] \quad \text{for each } n \geq 0$$

and in turn deduce that A^0_{cris} lies in $A_{\text{inf}}[[p^\flat]/p]$. □

PROPOSITION 3.1.12. Given a real number $\rho \in (0,1)$, the Frobenius automorphism of A_{inf} uniquely extends to a continuous injective endomorphism φ^+_ρ on B^+_ρ with $\varphi^+_\rho(B^+_\rho) \cong B^+_{\rho^p}$.

PROOF. The Frobenius automorphism of $A_{\text{inf}} = W(\mathcal{O}_F)$ uniquely extends to an automorphism on $A_{\text{inf}}[1/p]$, which we denote by φ_{inf}. Since we have

$$\varphi_{\text{inf}}\left(\sum [c_n]p^n\right) = \sum [c_n^p]p^n \quad \text{for each } c_n \in \mathcal{O}_F,$$

we obtain the equality

$$|\varphi_{\text{inf}}(f)|_{\rho^p} = |f|^p_\rho \quad \text{for each } f \in A_{\text{inf}}[1/p].$$

Hence Lemma 1.2.15 and Proposition 3.1.8 together show that φ_{inf} extends to a continuous injective homomorphism

$$\varphi^+_\rho : B^+_\rho \xrightarrow{\sim} B^+_{\rho^p} \hookrightarrow B^+_\rho$$

as desired. □

Definition 3.1.13. Given a real number $\rho \in (0,1)$, we refer to the map φ^+_ρ constructed in Proposition 3.1.12 as the *Frobenius endomorphism* on B^+_ρ and often write $\varphi = \varphi^+_\rho$.

PROPOSITION 3.1.14. The Frobenius endomorphism of B_{cris} is injective.

PROOF. Lemma 3.1.15 in Chapter III shows that the Frobenius endomorphism on $B^+_{1/p}$ restricts to the Frobenius endomorphism on B^+_{cris} via the natural embedding $B^+_{\text{cris}} \hookrightarrow B^+_{1/p}$ given by Proposition 3.1.11. Hence we deduce that the Frobenius endomorphism is injective on B^+_{cris}. Now we obtain the desired assertion as we have $B_{\text{cris}} = B^+_{\text{cris}}[1/t]$ for every cyclotomic uniformizer $t \in B^+_{\text{dR}}$. □

Remark. Proposition 3.1.14 establishes Proposition 3.2.1 in Chapter III. Our proof does not rely on any results which we stated without a proof.

Definition 3.1.15. We refer to the closure of $A_{\mathrm{inf}}[1/p]$ in B to be the *ring of Gauss normally integral elements in* B, denoted by B^+.

PROPOSITION 3.1.16. The ring B^+ is naturally a subring of B^+_{cris} with

$$B^+ = \bigcap_{n\geq 0} \varphi^n(B^+_{\mathrm{cris}}).$$

PROOF. Proposition 3.1.11 yields a natural injective homomorphism

$$B^+_{1/p^p} \hookrightarrow B^+_{\mathrm{cris}} \hookrightarrow B^+_{1/p}.$$

Moreover, Lemma 3.1.15 in Chapter III shows that the Frobenius endomorphism on $B^+_{1/p}$ restricts to the Frobenius endomorphism on B^+_{cris}. Hence we find

$$B^+_{1/p^{p^{n+1}}} = \varphi^n(B^+_{1/p^p}) \subseteq \varphi^n(B^+_{\mathrm{cris}}) \subseteq \varphi^n(B^+_{1/p}) = B^+_{1/p^{p^n}} \qquad \text{for every } n \geq 0$$

and in turn apply Proposition 3.1.8 to obtain the identity

$$B^+ = \bigcap_{\rho>0} B^+_\rho = \bigcap_{n\geq 0} B^+_{1/p^{p^n}} = \bigcap_{n\geq 0} \varphi^n(B^+_{\mathrm{cris}})$$

as desired. □

Remark. By Proposition 3.1.16, we may identify B^+ as the largest subring of B^+_{cris} on which φ is an automorphism.

PROPOSITION 3.1.17. The rings B^+_{cris}, B^+, and B satisfy the equality

$$(B^+_{\mathrm{cris}})^{\varphi=p^n} = (B^+)^{\varphi=p^n} = B^{\varphi=p^n} \qquad \text{for every } n \in \mathbb{Z}.$$

PROOF. It is not hard to see by Proposition 3.1.16 that $(B^+_{\mathrm{cris}})^{\varphi=p^n}$ and $(B^+)^{\varphi=p^n}$ coincide. In addition, $(B^+)^{\varphi=p^n}$ is evidently a subset of $B^{\varphi=p^n}$. Hence we only need to prove that $(B^+)^{\varphi=p^n}$ contains $B^{\varphi=p^n}$. If we have $n \leq 0$, the assertion is evident by Proposition 1.3.12 and Proposition 1.3.13. If we have $n \geq 1$, we find

$$\log(\varepsilon) = \sum_{m=1}^{\infty} (-1)^{m+1} \frac{([\varepsilon]-1)^m}{m} \in B^+ \qquad \text{for every } \varepsilon \in 1 + \mathfrak{m}_F,$$

as each summand lies in $A_{\mathrm{inf}}[1/p]$, and in turn apply Proposition 1.4.19 to establish the assertion. □

PROPOSITION 3.1.18. The scheme X admits an identification

$$X = \mathrm{Proj}\left(\bigoplus_{n\geq 0} (B^+_{\mathrm{cris}})^{\varphi=p^n}\right).$$

PROOF. The assertion is evident by Proposition 3.1.17. □

Remark. If we write $B_e^{(n)} := B_e \cap \mathrm{Fil}^{-n}(B_{\mathrm{dR}})$ for every $n \geq 0$, we can use Proposition 3.1.18 to obtain a natural isomorphism

$$X \cong \mathrm{Proj}\left(\bigoplus_{n\geq 0} B_e^{(n)}\right)$$

as described in Chapter I.

PROPOSITION 3.1.19. The ring $B_e = B_{\text{cris}}^{\varphi=1}$ is a principal ideal domain with a canonical isomorphism $B_e \cong B[1/t]^{\varphi=1}$ for every cyclotomic uniformizer $t \in B_{\text{dR}}^+$.

PROOF. Proposition 3.1.17 yields a natural identification

$$B_e = B_{\text{cris}}^+[1/t]^{\varphi=1} \cong B[1/t]^{\varphi=1}.$$

Hence we obtain a canonical isomorphism $U \cong \operatorname{Spec}(B_e)$ and in turn establish the desired assertion by Theorem 1.4.22. □

Remark. Fontaine originally deduced Proposition 3.1.19 from the result of Berger [**Ber08**], which shows that B_e is Bézout. As briefly mentioned in Chapter I, Fontaine's proof of Proposition 3.1.19 directly led to the constrution of the Fargues-Fontaine curve.

PROPOSITION 3.1.20. There exists a natural isomorphism $B_e^\times \cong \mathbb{Q}_p^\times$.

PROOF. Since we have $B_e \cong B[1/t]^{\varphi=1}$ for every cyclotomic uniformizer $t \in B_{\text{dR}}^+$ as noted in Proposition 3.1.19, the assertion is straightforward to verify by Proposition 1.4.19. □

THEOREM 3.1.21 (Fontaine [**Fon94a**]). The ring $B_e = B_{\text{cris}}^{\varphi=1}$ fits into a natural exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow B_e \longrightarrow B_{\text{dR}}/B_{\text{dR}}^+ \longrightarrow 0. \tag{3.4}$$

PROOF. Fix a cyclotomic uniformizer $t \in B_{\text{dR}}^+$. We assert that each $B^{\varphi=p^n}$ with $n \geq 1$ canonically fits into a short exact sequence

$$0 \longrightarrow \mathbb{Q}_p t^n \longrightarrow B^{\varphi=p^n} \longrightarrow B_{\text{dR}}^+/t^n B_{\text{dR}}^+ \longrightarrow 0. \tag{3.5}$$

Proposition 2.2.25 in Chapter III implies that both $\mathbb{Q}_p t^n$ and $B_{\text{dR}}^+/t^n B_{\text{dR}}^+$ do not depend on our choice of t. In addition, we apply Proposition 1.3.13 and Lemma 1.4.16 to identify $\mathbb{Q}_p t^n$ with the kernel of the natural embedding $B^{\varphi=p^n} \hookrightarrow B_{\text{dR}}^+$ given by Proposition 1.3.25. Hence it remains to prove that the map $B^{\varphi=p^n} \to B_{\text{dR}}^+/t^n B_{\text{dR}}^+$ is surjective. For $n = 1$, the assertion is evident by Proposition 1.4.3. Let us now proceed by induction on n. Take an arbitrary element $b \in B_{\text{dR}}^+$. Since $\mathbb{C}_K$ is algebraically closed by Proposition 3.1.10 in Chapter II, we use Proposition 1.4.3 to obtain an element $s \in B^{\varphi=p}$ with $b - s^n \in tB_{\text{dR}}^+$. By the induction hypothesis, we find $f \in B^{\varphi=p^{n-1}}$ and $b' \in B_{\text{dR}}^+$ with

$$b = s^n + t(f + t^{n-1}b') = (s^n + tf) + t^n b'.$$

We observe that $s^n + tf$ lies in $B^{\varphi=p^n}$ and thus deduce that the map $B^{\varphi=p^n} \to B_{\text{dR}}^+/t^n B_{\text{dR}}^+$ is surjective.

Our discussion in the preceding paragraph shows that for every $n \geq 1$ there exists a canonical short exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow t^{-n} B^{\varphi=p^n} \longrightarrow t^{-n} B_{\text{dR}}^+/B_{\text{dR}}^+ \longrightarrow 0.$$

Moreover, by Proposition 3.1.17 we have natural isomorphisms

$$B_e \cong B[1/t]^{\varphi=1} \cong \varinjlim t^{-n} B^{\varphi=p^n} \quad \text{and} \quad B_{\text{dR}}/B_{\text{dR}}^+ \cong \varinjlim t^{-n} B_{\text{dR}}^+/B_{\text{dR}}^+,$$

where the transition maps for the colimits are the canonical embeddings. Therefore we obtain the natural short exact sequence (3.4) as desired. □

Remark. Theorem 3.1.21 establishes Theorem 3.2.19 in Chapter III. Our proof invokes Lemma 1.4.16 and thus relies on Proposition 1.4.14, which we stated without a proof. We note that the exact sequence (3.5) coincides with the short exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X(n)) \longrightarrow H^0(X, \mathcal{O}_X(n)/t^n\mathcal{O}_X) \longrightarrow 0,$$

where $t^n\mathcal{O}_X$ denotes the quasicoherent $\mathcal{O}_X$-module associated to $t^n P$.

3.2. Geometrization of isocrystals

Throughout this subsection, we fix a cyclotomic uniformizer $t \in B_{\mathrm{dR}}^+$ and generally write φ for a map naturally induced by the Frobenius endomorphism on B_{cris} or an isocrystal. In addition, we denote the completed stalk of an $\mathcal{O}_X$-module $\mathcal{V}$ at ∞ by $\widehat{\mathcal{V}_\infty}$. Let us state the following generalization of Proposition 3.1.17 without a proof.

Proposition 3.2.1. Given integers d and r with $r > 0$, we have

$$(B_{\mathrm{cris}}^+)^{\varphi^r = p^d} = (B^+)^{\varphi^r = p^d} = B^{\varphi^r = p^d}.$$

Remark. The first equality is not hard to verify by Proposition 3.1.16. The second equality follows from the fact that the map $\log_r$ described in the remark after Proposition 2.3.5 takes values in $(B^+)^{\varphi^r = p}$, as explained in the article of Fargues-Fontaine [**FF18**, §6.3] or the notes of Lurie [**Lur**, Lectures 24].

Definition 3.2.2. Given an isocrystal D over K_0, its *crystalline $\mathcal{O}_X$-module* is the quasicoherent $\mathcal{O}_X$-module $\mathcal{E}(D)$ associated to $P(D) := \bigoplus_{n \geq 0} (D \otimes_{K_0} B)^{\varphi = p^n}$.

Lemma 3.2.3. Let r and d be integers with $r > 0$. For every homogeneous $f \in P$, the $B[1/f]^{\varphi=1}$-module $B[1/f]^{\varphi^r = p^d}$ is free of rank r with a basis $(\varphi^i(g))$ for some $g \in B[1/f]^{\varphi^r = p^d}$.

Proof. By Lemma 2.3.7, there exists a basis element g of $B[1/f]^{\varphi^r = p^{-d}}$ over $B[1/f]^{\varphi^r = 1}$. Moreover, Lemma 2.3.2 yields a canonical isomorphism

$$B[1/f]^{\varphi=1} \otimes_{\mathbb{Q}_p} E_r \cong B[1/f]^{\varphi^r=1}$$

Hence we deduce that $B[1/f]^{\varphi^r = p^{-d}}$ admits a basis $(\varphi^i(g))$ over $B[1/f]^{\varphi=1}$. □

Proposition 3.2.4. Let D be an isocrystal over K_0 of rank r and degree d.

(1) The $\mathcal{O}_X$-module $\mathcal{E}(D)$ is a vector bundle on X of rank r and degree $-d$.
(2) There exist natural isomorphisms

$$H^0(U, \mathcal{E}(D)) \cong (D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \quad \text{and} \quad \widehat{\mathcal{E}(D)_\infty} \cong D_K \otimes_K B_{\mathrm{dR}}^+.$$

Proof. Since $\widehat{K_0^{\mathrm{un}}} = W(\overline{k})[1/p]$ naturally embeds into B and B_{cris}, we may replace D by $D \otimes_{K_0} \widehat{K_0^{\mathrm{un}}}$ to assume that k is algebraically closed. By Theorem 2.3.25 in Chapter II, we may further assume that D is isomorphic to a simple isocrystal D_λ with $\lambda = d/r$. Let us choose a K_0-basis $(\varphi^i(e))$ of $D \simeq D_\lambda$ with $e \in D$ and $\varphi^r(e) = p^d e$. For every nonzero homogeneous element $f \in P$, the open subscheme $U(f) := \mathrm{Spec}\,(B[1/f]^{\varphi=1})$ of $X = \mathrm{Proj}\,(P)$ yields a canonical isomorphism

$$\mathcal{E}(D)(U(f)) \cong (D \otimes_{K_0} B[1/f])^{\varphi=1} \cong B[1/f]^{\varphi^r = p^{-d}}.$$

Hence we find $\mathcal{E}(D) \cong \mathcal{O}(-\lambda)$ and in turn establish statement (1).

It remains to prove statement (2). By Proposition 3.2.1, we have a natural isomorphism

$$H^0(U, \mathcal{E}(D)) \cong B[1/t]^{\varphi^r = p^{-d}} \cong B_{\mathrm{cris}}^+[1/t]^{\varphi^r = p^{-d}} \cong (D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1}.$$

Moreover, if we take a nonzero homogeneous element $f \in P$ with $\infty \in U(f)$, we apply Proposition 3.1.5 and Lemma 3.2.3 to obtain an identification

$$\widehat{\mathcal{E}(D)_\infty} \cong \mathcal{E}(D)(U(f)) \otimes_{B[1/f]^{\varphi=1}} B_{\mathrm{dR}}^+ \cong B[1/f]^{\varphi^r = p^{-d}} \otimes_{B[1/f]^{\varphi=1}} B_{\mathrm{dR}}^+ \cong D_K \otimes_K B_{\mathrm{dR}}^+.$$

Therefore we establish the desired assertion. □

Definition 3.2.5. A (B_e, B_{dR}^+)-*pair* is a pair (M_e, M_{dR}^+) consisting of a free B_e-module M_e of finite rank and a B_{dR}^+-lattice M_{dR}^+ in $M_e \otimes_{B_e} B_{\mathrm{dR}}$.

PROPOSITION 3.2.6. There exists a equivalence of categories

$$\{\, (B_e, B_{\mathrm{dR}}^+)\text{-pairs} \,\} \xrightarrow{\sim} \{\text{ vector bundles on } X \,\}$$

which maps each (B_e, B_{dR}^+)-pair (M_e, M_{dR}^+) to a vector bundle $\mathcal{V}$ on X with identifications

$$H^0(U, \mathcal{V}) \cong M_e, \quad \widehat{\mathcal{V}_\infty} \cong M_{\mathrm{dR}}^+, \quad H^0(X, \mathcal{V}) \cong M_e \cap M_{\mathrm{dR}}^+.$$

PROOF. Proposition 3.1.5 and Proposition 3.1.19 respectively yield natural isomorphisms

$$\widehat{\mathcal{O}_{X,\infty}} \cong B_{\mathrm{dR}}^+ \quad \text{and} \quad U \cong \mathrm{Spec}\,(B_e).$$

Moreover, by Theorem 1.4.22, the closed point ∞ lies in an open subscheme of X which is the spectrum of a principal ideal domain. Hence the desired assertion is straightforward to verify by the theorem of Beauville-Laszlo [**BL95**] stated in the Stacks project [**Sta**, Tag 0BP2]. □

PROPOSITION 3.2.7. Every filtered isocrystal D over K naturally yields a vector bundle $\mathcal{F}(D)$ on X with identifications

$$H^0(U, \mathcal{F}(D)) \cong (D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \quad \text{and} \quad \widehat{\mathcal{F}(D)_\infty} \cong \mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}}).$$

PROOF. Proposition 3.2.4 and Proposition 3.2.6 together yield an isomorphism

$$(D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \otimes_{B_e} B_{\mathrm{dR}} \cong D_K \otimes_K B_{\mathrm{dR}}.$$

Meanwhile, we see by Proposition 2.3.6 in Chapter III that $\mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}})$ is a B_{dR}^+-lattice in $D_K \otimes_K B_{\mathrm{dR}}$. Hence we deduce the desired assertion from Proposition 3.2.6. □

Definition 3.2.8. Given a filtered isocrystal D over K, we refer to the vector bundle $\mathcal{F}(D)$ on X given by Proposition 3.2.7 as the *modified crystalline $\mathcal{O}_X$-module* of D.

LEMMA 3.2.9. Given a line bundle $\mathcal{L}$ on X and a coherent nonzero $\mathcal{O}_X$-submodule $\mathcal{M}$ of $\mathcal{L}$ with the quotient $\mathcal{L}/\mathcal{M}$ supported at ∞, there exists a natural isomorphism

$$\widehat{\mathcal{L}_\infty}/\widehat{\mathcal{M}_\infty} \cong B_{\mathrm{dR}}^+/t^{\deg(\mathcal{L})-\deg(\mathcal{M})}B_{\mathrm{dR}}^+.$$

PROOF. The assertion is straightforward to verify. □

PROPOSITION 3.2.10. Every filtered isocrystal D over K satisfies the equalities

$$\mathrm{rk}(\mathcal{F}(D)) = \mathrm{rk}(D) \quad \text{and} \quad \deg(\mathcal{F}(D)) = \deg^\bullet(D) - \deg(D).$$

PROOF. The first equality is evident by Proposition 3.2.4 and Proposition 3.2.7. Hence it remains to establish the second equality. Let us take the Hodge-Tate weights $m_1, \cdots, m_r$ of D in ascending order and write $m := \min(0, m_1)$. We can construct a K-basis $(v_{i,j})$ of D_K such that each $\mathrm{Fil}^{m_s}(D_K)$ has a K-basis $(v_{i,j})_{i \geq s}$; indeed, we choose a K-basis for $\mathrm{Fil}^{m_r}(D_K)$ and inductively extend a K-basis for each $\mathrm{Fil}^{m_s}(D_K)$ to a K-basis for $\mathrm{Fil}^{m_{s-1}}(D_K)$. For every $n \in \mathbb{Z}$, we deduce from Proposition 2.3.6 in Chapter III that $\mathrm{Fil}^0(D(n)_K \otimes_K B_{\mathrm{dR}})$ admits a B_{dR}^+-basis $(v_{i,j} \otimes t^{n-m_i})$. Therefore Proposition 3.2.4 and Proposition 3.2.7 together imply that $\mathcal{E}(D)$ and $\mathcal{F}(D)$ are naturally coherent $\mathcal{O}_X$-submodules of $\mathcal{F}(D(m))$ with their quotients supported at ∞. Moreover, we apply Proposition 3.3.9 in Chapter III to find

$$\det(\widehat{\mathcal{F}(D(m))})_\infty / \det(\widehat{\mathcal{E}(D)})_\infty \cong t^{m \cdot \mathrm{rk}(D) - \deg^\bullet(D)} B_{\mathrm{dR}}^+/B_{\mathrm{dR}}^+,$$

$$\det(\widehat{\mathcal{F}(D(m))})_\infty / \det(\widehat{\mathcal{F}(D)})_\infty \cong t^{m \cdot \mathrm{rk}(D)} B_{\mathrm{dR}}^+/B_{\mathrm{dR}}^+.$$

Now the assertion is straightforward to verify by Proposition 3.2.4 and Lemma 3.2.9. □

Proposition 3.2.11. The Γ_K-action on B_{dR} naturally induces a Γ_K-action on X, under which both U and ∞ are stable.

Proof. We note that φ is Γ_K-equivariant on B_{cris}^+ and in turn find that each $(B_{\mathrm{cris}}^+)^{\varphi=p^n}$ is stable under the Γ_K-action on B_{dR}. Hence Proposition 3.1.18 implies that the Γ_K-action on B_{dR} naturally gives rise to a Γ_K-action on X. Moreover, both $U = X\setminus\{\infty\}$ and ∞ are stable under the Γ_K-action on X as Γ_K acts on t via the cyclotomic character of K by Theorem 2.2.26 in Chapter III. □

Definition 3.2.12. A Γ_K*-equivariant vector bundle* on X is a vector bundle $\mathcal{V}$ on X with an isomorphism $c_\gamma : \gamma^*\mathcal{V} \simeq \mathcal{V}$ for each $\gamma \in \Gamma_K$, called a γ*-twist map*, satisfying the relation

$$c_{\gamma\gamma'} = c_{\gamma'} \circ (\gamma')^*(c_\gamma) \quad \text{for every } \gamma, \gamma' \in \Gamma_K.$$

Proposition 3.2.13. Let D be a filtered isocrystal over K.

(1) The $\mathcal{O}_X$-module $\mathcal{F}(D)$ is naturally a Γ_K-equivariant vector bundle on X.
(2) The $\mathbb{Q}_p$-vector space $H^0(X, \mathcal{F}(D))$ admits a natural Γ_K-action with an identification

$$H^0(X, \mathcal{F}(D)) = (D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \cap \mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}}).$$

Proof. Theorem 2.2.26 in Chapter III shows that each $\mathrm{Fil}^n(B_{\mathrm{dR}}) = t^n B_{\mathrm{dR}}^+$ is stable under the Γ_K-action on B_{dR}. Moreover, φ is Γ_K-equivariant on B_{cris} by construction. Hence we see that Γ_K naturally acts on $H^0(U, \mathcal{F}(D)) \cong (D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1}$ and $\widehat{\mathcal{F}(D)_\infty} \cong \mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}})$. Now the desired assertions are straightforward to verify by Proposition 3.2.6. □

Lemma 3.2.14. Every vector bundle $\mathcal{V}$ on X satisfies the equalities

$$\mathrm{rk}(\gamma^*\mathcal{V}) = \mathrm{rk}(\mathcal{V}) \quad \text{and} \quad \deg(\gamma^*\mathcal{V}) = \deg(\mathcal{V}) \quad \text{for each } \gamma \in \Gamma_K.$$

Proof. Since the first equality is evident, we only need to establish the second equality. Let us write $d := \deg(\mathcal{V})$. We apply Proposition 2.1.6 to see that $\det(\mathcal{V})$ is isomorphic to the line bundle give by the Weil devisor $d\infty$ on X. Moreover, we deduce from Proposition 3.2.11 that $\det(\gamma^*\mathcal{V}) \cong \gamma^* \det(\mathcal{V})$ is also isomorphic to the line bundle give by the Weil devisor $d\infty$ on X. Hence we obtain the second equality as desired, thereby completing the proof. □

Lemma 3.2.15. Given an element $\gamma \in \Gamma_K$, a vector bundle $\mathcal{V}$ on X is semistable if and only if $\gamma^*\mathcal{V}$ is semistable.

Proof. There exists a natural bijection

$$\{\text{ subbundles of } \mathcal{V}\} \xrightarrow{\sim} \{\text{ subbundles of } \gamma^*\mathcal{V}\}$$

which sends each subbundle $\mathcal{W}$ of $\mathcal{V}$ to $\gamma^*\mathcal{W}$. Hence the assertion is an immediate consequence of Lemma 3.2.14. □

Proposition 3.2.16. Given a Γ_K-equivariant vector bundle $\mathcal{V}$ on X with a Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V},$$

each $\mathcal{V}_i$ is naturally a Γ_K-equivariant vector bundle on X.

Proof. For every $\gamma \in \Gamma_K$, we apply Lemma 3.2.14 and Lemma 3.2.15 to see that the vector bundle $\gamma^*\mathcal{V}$ on X admits a Harder-Narsimhan filtration

$$0 = \gamma^*\mathcal{V}_0 \subset \gamma^*\mathcal{V}_1 \subset \cdots \subset \gamma^*\mathcal{V}_n = \gamma^*\mathcal{V}.$$

Since $\mathcal{V}$ admits a unique Harder-Narasimhan filtration by Theorem 2.2.20, we deduce that each $\mathcal{V}_i$ is a Γ_K-equivariant vector bundle with a γ-twist map $c_{\gamma,i} : \gamma^*\mathcal{V}_i \simeq \mathcal{V}_i$ given by the γ-twist map $c_\gamma : \gamma^*\mathcal{V} \simeq \mathcal{V}$ for $\mathcal{V}$. □

Proposition 3.2.17. Let $\mathcal{V}$ be a Γ_K-equivariant vector bundle on X.

(1) The K_0-vector space $D(\mathcal{V}) := (H^0(U, \mathcal{V}) \otimes_{B_e} B_{\text{cris}})^{\Gamma_K}$ is naturally an isocrystal and gives rise to a canonical injective B_{cris}-linear Γ_K-equivariant map
$$\alpha_{\mathcal{V}} : D(\mathcal{V}) \otimes_{K_0} B_{\text{cris}} \hookrightarrow H^0(U, \mathcal{V}) \otimes_{B_e} B_{\text{cris}}.$$
(2) $D(\mathcal{V})$ satisfies the inequality
$$\text{rk}(D(\mathcal{V})) \leq \text{rk}(\mathcal{V}) \tag{3.6}$$
with equality precisely when $\alpha_{\mathcal{V}}$ is an isomorphism.

Proof. Let us consider the natural B_{cris}-linear Γ_K-equivariant map
$$\alpha_{\mathcal{V}} : D(\mathcal{V}) \otimes_{K_0} B_{\text{cris}} \longrightarrow H^0(U, \mathcal{V}) \otimes_{B_e} B_{\text{cris}} \otimes_{B_e} B_{\text{cris}} \longrightarrow H^0(U, \mathcal{V}) \otimes_{B_e} B_{\text{cris}}$$
We assert that $\alpha_{\mathcal{V}}$ is injective. Denote by C_{cris} the fraction field of B_{cris}. It suffices to prove the injectivity of the induced C_{cris}-linear map
$$\beta_{\mathcal{V}} : D(\mathcal{V}) \otimes_{K_0} C_{\text{cris}} \longrightarrow H^0(U, \mathcal{V}) \otimes_{B_e} C_{\text{cris}}.$$
Suppose for contradiction that $\ker(\beta_{\mathcal{V}})$ is not trivial. Take a K_0-basis (e_i) of $D(\mathcal{V})$ and choose a nontrivial C_{cris}-linear relation $\sum c_i e_i = 0$ with minimal number of nonzero terms. We may set $c_j = 1$ for some j. For every $\gamma \in \Gamma_K$, we find
$$\sum (\gamma(c_i) - c_i) e_i = \gamma\left(\sum c_i e_i\right) - \sum c_i e_i = 0 \quad \text{and} \quad \gamma(c_j) - c_j = \gamma(1) - 1 = 0.$$
By the minimality of our relation, each c_i satisfies the equality $c_i = \gamma(c_i)$ for every $\gamma \in \Gamma_K$. Hence Theorem 3.1.14 in Chapter III shows that each c_i lies in $C_{\text{cris}}^{\Gamma_K} \cong K_0$. Now we have a nontrivial K_0-linear relation $\sum c_i e_i = 0$ for the K_0-basis (e_i) of $D(\mathcal{V})$, thereby obtaining a desired contradiction.

The injectivity of $\alpha_{\mathcal{V}}$ implies that the K_0-vector space $D(\mathcal{V})$ is finite dimensional. Meanwhile, for arbitrary $v \in H^0(U, \mathcal{V})$, $b \in B_{\text{cris}}$, and $c \in K_0$, we find
$$(1 \otimes \varphi)(c(v \otimes b)) = (1 \otimes \varphi)(v \otimes bc) = v \otimes \varphi(b)\varphi(c) = \varphi(c) \cdot (1 \otimes \varphi)(v \otimes b).$$
Since φ extends the Frobenius automorphism σ of K_0, the additive map $1 \otimes \varphi$ is σ-semilinear. In addition, the map $1 \otimes \varphi$ is injective on $D(\mathcal{V})$ by Proposition 3.1.14. Hence we deduce from Lemma 3.2.5 in Chapter III that $D(\mathcal{V})$ is an isocrystal over K_0 with Frobenius automorphism $1 \otimes \varphi$ and in turn establish statement (1).

It remains to verify statement (2). Since the inequality (3.6) is evident by statement (1), we only need to consider the equality condition. If $\alpha_{\mathcal{V}}$ is an isomorphism, the inequality becomes an equality. For the converse, we henceforth assume the identity $\text{rk}(D(\mathcal{V})) = \text{rk}(\mathcal{V})$. Let us choose a K_0-basis (u_i) of $D(\mathcal{V})$ and a B_e-basis (v_i) of $H^0(U, \mathcal{V})$. We may represent $\alpha_{\mathcal{V}}$ by a $r \times r$ matrix $M_{\mathcal{V}}$ with $r := \text{rk}(D(\mathcal{V})) = \text{rk}(\mathcal{V})$. We wish to show that $\det(M_{\mathcal{V}})$ is a unit in B_{cris}. We have $\det(M_{\mathcal{V}}) \neq 0$ as the C_{cris}-linear map $\beta_{\mathcal{V}}$ induced by $\alpha_{\mathcal{V}}$ is an isomorphism for being an injective map between vector spaces of equal dimension. Meanwhile, Γ_K acts trivially on $u_1 \wedge \cdots \wedge u_d$ and by some B_e-valued character η on $v_1 \wedge \cdots \wedge v_d$. Since the Γ_K-equivariant map $\alpha_{\mathcal{V}}$ yields the identity
$$(\wedge^d \alpha_{\mathcal{V}})(u_1 \wedge \cdots \wedge u_d) = \det(M_{\mathcal{V}})(v_1 \wedge \cdots \wedge v_d),$$
we deduce that Γ_K acts on $\det(M_{\mathcal{V}})$ by η^{-1}. Now we observe by Proposition 3.1.20 that η is $\mathbb{Q}_p$-valued and in turn find by Theorem 3.1.14 in Chapter III that $\det(M_{\mathcal{V}})$ is a unit in B_{cris}, thereby completing the proof. □

Definition 3.2.18. Given a Γ_K-equivariant vector bundle $\mathcal{V}$ on X, its *associated isocrystal* is the isocrystal $D(\mathcal{V})$ constructed in Proposition 3.2.17..

Lemma 3.2.19. Every isocrystal D over K_0 admits a natural isomorphism

$$D \cong ((D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \otimes_{B_e} B_{\mathrm{cris}})^{\Gamma_K}.$$

Proof. Since $\widehat{K_0^{\mathrm{un}}}$ naturally embeds into B_{cris}, we may replace D by $D \otimes_{K_0} \widehat{K_0^{\mathrm{un}}}$ to assume that k is algebraically closed. By Theorem 2.3.25 in Chapter II, we may further assume that D is isomorphic to a simple isocrystal D_λ. Let us write $\lambda = d/r$ in reduced form and choose a K_0-basis $(\varphi^i(e))$ of $D \simeq D_\lambda$ with $e \in D$ and $\varphi^r(e) = p^d e$. We have an identification

$$(D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \cong B_{\mathrm{cris}}^{\varphi^r = p^{-d}}.$$

In addition, Proposition 3.2.1 and Lemma 3.2.3 together show that $B_{\mathrm{cris}}^{\varphi^r=p^{-d}} = B_{\mathrm{cris}}^+[1/t]^{\varphi^r=p^{-d}}$ admits a basis $(\varphi^i(g))$ over $B_e = B_{\mathrm{cris}}^+[1/t]^{\varphi=1}$ for some $g \in B_{\mathrm{cris}}^{\varphi^r=p^{-d}}$. Hence we obtain a canonical isomorphism

$$(D \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1} \otimes_{B_e} B_{\mathrm{cris}} \cong B_{\mathrm{cris}}^{\varphi^r=p^{-d}} \otimes_{B_e} B_{\mathrm{cris}} \cong D \otimes_{K_0} B_{\mathrm{cris}}.$$

It is straightforward to verify that this map is Γ_K-equivariant. Now we deduce the desired assertion from Theorem 3.1.14 in Chapter III. □

Proposition 3.2.20. Given a weakly admissible filtered isocrystal D over K, the vector bundle $\mathcal{F}(D)$ on X is trivial with $\mathrm{rk}(\mathcal{F}(D)) = \mathrm{rk}(D)$.

Proof. Let us write $\mathcal{V} := \mathcal{F}(D)$ for notational convenience. By Proposition 3.2.10, we find $\mathrm{rk}(\mathcal{F}(D)) = \mathrm{rk}(D)$ and $\deg(\mathcal{F}(D)) = 0$. In light of Proposition 2.3.17, it suffices to show that $\mathcal{V}$ is semistable. Suppose for contradiction that $\mathcal{V}$ is not semistable. Theorem 2.2.20 yields a Harder-Narasimhan filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n = \mathcal{V} \quad \text{with } n > 1.$$

Let us set $\mathcal{V}' := \mathcal{V}_1$ and $\mathcal{V}'' := \mathcal{V}/\mathcal{V}'$. We find $\mu(\mathcal{V}') > \mu(\mathcal{V}) = 0$ by Proposition 2.2.8. In addition, Proposition 3.2.16 implies that both $\mathcal{V}'$ and $\mathcal{V}''$ are naturally Γ_K-equivariant vector bundles on X. Hence we have a short exact sequence

$$0 \longrightarrow D(\mathcal{V}') \longrightarrow D(\mathcal{V}) \longrightarrow D(\mathcal{V}'').$$

Since Proposition 3.2.17 yields the inequalities

$$\mathrm{rk}(D(\mathcal{V}')) \leq \mathrm{rk}(\mathcal{V}') \quad \text{and} \quad \mathrm{rk}(D(\mathcal{V}'')) \leq \mathrm{rk}(\mathcal{V}''),$$

we obtain the relation

$$\mathrm{rk}(D(\mathcal{V})) \leq \mathrm{rk}(D(\mathcal{V}')) + \mathrm{rk}(D(\mathcal{V}'')) \leq \mathrm{rk}(\mathcal{V}') + \mathrm{rk}(\mathcal{V}'') = \mathrm{rk}(\mathcal{V}).$$

Meanwhile, we see that D admits a natural isomorphism $D \cong D(\mathcal{V})$ by Lemma 3.2.19 and in turn find $\mathrm{rk}(D(\mathcal{V})) = \mathrm{rk}(\mathcal{V})$ by Proposition 3.2.10. Therefore all nonstrict inequalities are in fact equalities. Now Proposition 3.2.17 yields a canonical isomorphism

$$H^0(U, \mathcal{V}') \cong (H^0(U, \mathcal{V}') \otimes_{B_e} B_{\mathrm{cris}})^{\varphi=1} \cong (D(\mathcal{V}') \otimes_{K_0} B_{\mathrm{cris}})^{\varphi=1}. \tag{3.7}$$

Take D' to be the isocrystal $D(\mathcal{V}')$ with the filtration on D'_K given by

$$\mathrm{Fil}^n(D'_K) := \mathrm{Fil}^n(D_K) \cap D'_K \quad \text{for each } n \in \mathbb{Z},$$

where we regard $D(\mathcal{V}')$ as a subisocrystal of $D \cong D(\mathcal{V})$. We obtain a natural isomorphism

$$\widehat{\mathcal{V}'_\infty} \cong \mathrm{Fil}^0(D'_K \otimes_K B_{\mathrm{dR}}).$$

and in turn identify $\mathcal{V}'$ with $\mathcal{F}(D')$. Hence we apply Proposition 3.2.10 to find

$$\mu(\mathcal{V}') = \deg^\bullet(D') - \deg(D') \leq 0$$

as D is weakly admissible, thereby obtaining a desired contradiction. □

PROPOSITION 3.2.21. Given a weakly admissible filtered isocrystal D over K, the $\mathbb{Q}_p$-vector space $H^0(X, \mathcal{F}(D))$ is naturally a p-adic Γ_K-representation with

$$\dim_{\mathbb{Q}_p} H^0(X, \mathcal{F}(D)) = \mathrm{rk}(D). \tag{3.8}$$

PROOF. Since we have $H^0(X, \mathcal{O}_X) \cong \mathbb{Q}_p$ by Proposition 1.3.13 and Theorem 2.1.8, we obtain the equality (3.8) by Proposition 3.2.20. Moreover, we deduce from Proposition 3.1.2 that the Γ_K-action on $H^0(X, \mathcal{F}(D))$ given by Proposition 3.2.13 is continuous with respect to the p-adic topology, thereby completing the proof. □

THEOREM 3.2.22 (Colmez-Fontaine [**CF00**]). A filtered isocrystal D over K is admissible if and only if it is weakly admissible.

PROOF. If D is admissible, it is weakly admissible by Proposition 3.3.3 in Chapter III. For the converse, we now assume that D is weakly admissible and write $V := H^0(X, \mathcal{F}(D))$. Proposition 3.2.13 yields a natural B_{cris}-linear map

$$\alpha_V : V \otimes_{\mathbb{Q}_p} B_{\text{cris}} \longrightarrow D \otimes_{K_0} B_{\text{cris}} \otimes_{\mathbb{Q}_p} B_{\text{cris}} \longrightarrow D \otimes_{K_0} B_{\text{cris}}$$

which is compatible with the Γ_K-actions and the Frobenius endomorhpisms. Let us denote the fraction field of B_{cris} by C_{cris} and concider the induced C_{cris}-linear map

$$\beta_V : V \otimes_{\mathbb{Q}_p} C_{\text{cris}} \longrightarrow D \otimes_{K_0} C_{\text{cris}}.$$

We note that $\beta_V(V \otimes_{\mathbb{Q}_p} C_{\text{cris}})$ is stable under the Γ_K-action and thus apply Theorem 3.1.14 in Chapter III to obtain a K_0-subspace D' of D with $\beta_V(V \otimes_{\mathbb{Q}_p} C_{\text{cris}}) = D' \otimes_{K_0} C_{\text{cris}}$; indeed, if we identify $\beta_V(V \otimes_{\mathbb{Q}_p} C_{\text{cris}})$ as a C_{cris}-point of a Grassmannian for D, it descends to a K_0-point for being Γ_K-invariant. Moreover, D' is naturally a filtered subisocrystal of D with

$$\mathrm{Fil}^n(D'_K) := \mathrm{Fil}^n(D_K) \cap D'_K \quad \text{for each } n \in \mathbb{Z}.$$

Since β_V is injective on V by construction, we find

$$V \subseteq (D \otimes_{K_0} B_{\text{cris}}) \cap (D' \otimes_{K_0} C_{\text{cris}}) = D' \otimes_{K_0} B_{\text{cris}}$$

and in turn establish the identification $V = H^0(X, \mathcal{F}(D'))$ by Proposition 3.2.13.

Choose a K_0-basis (e_i) of D' and a C_{cris}-basis $(\beta_V(v_j))$ of $\beta_V(V \otimes_{\mathbb{Q}_p} C_{\text{cris}})$ with $v_j \in V$. Each $\beta_V(v_j)$ admits an identity $\beta_V(v_j) = \sum b_{i,j} e_i$ with $b_{i,j} \in B_{\text{cris}}$. Let us write $r := \mathrm{rk}(D')$ and denote by M the $r \times r$ matrix whose (i,j)-entry is $b_{i,j}$. We find $\det(M) \neq 0$ by observing that M represents a C_{cris}-linear isomorphism. In addition, we obtain the equality

$$(\wedge^r \beta_V)(v_1 \wedge \cdots \wedge v_r) = \det(M)(e_1 \wedge \cdots \wedge e_r) \tag{3.9}$$

and consequently use Proposition 3.2.13 to identify $(\wedge^r \beta_V)(v_1 \wedge \cdots \wedge v_r)$ as a nonzero element of $H^0(X, \mathcal{F}(\det(D')))$. Since Proposition 3.2.10 shows that $\mathcal{F}(\det(D'))$ is a line bundle with

$$\deg(\mathcal{F}(\det(D'))) = \deg^\bullet(D') - \deg(D') \leq 0,$$

we have $\deg^\bullet(D') = \deg(D')$ by Proposition 1.3.12 and Theorem 2.1.8. Hence we deduce that D' is weakly admissible. Now Proposition 3.2.21 yields the identity $\mathrm{rk}(D') = \mathrm{rk}(D)$, which implies that D' and D coincide. We see that β_V is an isomorphism for being a surjective map between vector spaces of equal dimension. Meanwhile, since Γ_K acts trivially on $e_1 \wedge \cdots \wedge e_r$ and via some $\mathbb{Q}_p$-valued character η on $v_1 \wedge \cdots \wedge v_r$, we note by the equality (3.9) that Γ_K acts on $\det(M)$ via η^{-1} and in turn find $\det(M) \in B_{\text{cris}}^\times$ by Theorem 3.1.14 in Chapter III. Therefore α_V is an isomorphism and gives rise to an identification $D \cong D_{\text{cris}}(V)$. Now we observe by Proposition 3.2.21 that V is crystalline, thereby deducing that D is admissible as desired. □

Exercises

1. Let ξ be an element in A_{inf}.
   (1) Show that an element $\xi \in A_{\text{inf}}$ is primitive if and only if it satisfies the following equivalent conditions:
      (i) ξ is a unit multiple of $[m] - p$ for some $m \in \mathfrak{m}_F$.
      (ii) ξ generates $\ker(\theta_C)$ for some untilt C of F.
   (2) If ξ is primitive, show that ξA_{inf} contains infinitely many primitive elements.

2. Let ξ be a primitive element in A_{inf}.
   (1) Show that ξ is irreducible in A_{inf}.
   (2) Show that every $f \in A_{\text{inf}}$ admits an identity $f = g\xi + [c]$ with $g \in A_{\text{inf}}$ and $c \in \mathcal{O}_F$.

3. Given a primitive element $\xi \in A_{\text{inf}}$, prove that A_{inf} is ξ-adically complete.

Hint. Prove that each $A_{\text{inf}}/p^n A_{\text{inf}}$ is ξ-adically complete.

4. Let f be an element in $A_{\text{inf}}[1/p, 1/[\varpi]]$ with a Teichmüller expansion $f = \sum [c_n]p^n$.
   (1) Show that the Newton polygon of f, given by the largest decreasing convex function $\mathcal{N}_f : \mathbb{R} \to \mathbb{R} \cup \{\, \infty \,\}$ with $\mathcal{N}_f(n) \leq \nu_F(c_n)$ for each $n \in \mathbb{Z}$, yields the equality
   $$\mathcal{L}_f(s) = \inf_{r \in \mathbb{R}} (\mathcal{N}_f(r) + rs) \quad \text{ for each } s \in (0, \infty).$$
   (2) Show that an integer n is a slope for the piecewise linear function $\mathcal{L}_f$ if and only if $\mathcal{N}_f$ is not differentiable at n.
   (3) Show that a rational number λ is a slope for the piecewise linear function $\mathcal{N}_f$ if and only if $\mathcal{L}_f$ is not differentiable at n.

5. Let f be a nonzero element in A_{inf}.
   (1) Show that f is a unit if and only if $\mathcal{L}_f$ is the zero function on $(0, \infty)$.
   (2) Show that f is strongly primitive if and only if there exists $r \in (0, \infty)$ with
   $$\mathcal{L}_f(s) = \begin{cases} s & \text{for } s \leq r, \\ r & \text{for } s \geq r. \end{cases}$$

6. Given a closed interval $[a, b] \subseteq (0, 1)$, let y be an element in $Y_{[a,b]}$ and C be its representative.
   (1) Prove that the map θ_C uniquely extends to a surjective continuous open ring homomorphism $\widehat{\theta_C^{a,b}} : B_{[a,b]} \twoheadrightarrow C$.
   (2) Prove that there exists a natural isomorphism
   $$B_{\text{dR}}^+(y) \cong \varprojlim_i B_{[a,b]} / \ker(\widehat{\theta_C^{a,b}})^i.$$

7. Let n be a positive integer.

(1) Show that the $\mathbb{Q}_p$-vector space $A_{\mathrm{inf}}[1/p, 1/[\varpi]]^{\varphi=p^n}$ vanishes.

(2) Show that the $\mathbb{Q}_p$-vector space $B^{\varphi=p^n}$ is infinite dimensional.

8. Given an element $y \in Y$ and a nonzero element $t \in B^{\varphi=p}$ vanishing at y, establish a natural isomorphism

$$H^1(X, \mathcal{O}(d)) \cong B^+_{\mathrm{dR}}(y)/(t^{-d}B^+_{\mathrm{dR}}(y) + \mathbb{Q}_p) \quad \text{for every } d < 0.$$

Remark. For $d = -1$, we see that $H^1(X, \mathcal{O}(-1))$ does not vanish whereas $H^1(\mathbb{P}^1_{\mathbb{C}}, \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(-1))$ vanishes.

9. Let $\mathcal{V}$ be a nonzero vector bundle on a complete abstract curve Z and write $\mathcal{V}^\vee$ for the dual bundle of $\mathcal{V}$.

(1) Verify the equality $\mu(\mathcal{V}^\vee) = -\mu(\mathcal{V})$.

(2) Show that $\mathcal{V}$ is semistable if and only if $\mathcal{V}^\vee$ is semistable.

10. Given a complete abstract curve Z and a rational number λ, show that the zero bundle and the semistable vector bundles of slope λ on Z together form an abelian category.

11. Let Z be a complete abstract curve.

(1) If the degree map $\deg_Z$ is not an isomorphism, prove that there exist semistable vector bundles $\mathcal{V}$ and $\mathcal{W}$ on Z with $\mu(\mathcal{V}) \leq \mu(\mathcal{W})$ and $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{V}, \mathcal{W}) = 0$.

(2) If Z is either $\mathbb{P}^1_{\mathbb{C}}$ or X, for arbitrary semistable vector bundles $\mathcal{V}$ and $\mathcal{W}$ on Z with $\mu(\mathcal{V}) \leq \mu(\mathcal{W})$ prove that $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{V}, \mathcal{W})$ does not vanish.

12. Let Z be a complete abstract curve.

(1) Prove that the tensor product of a semistable vector bundle and a line bundle on Z is semistable.

(2) If Z is either $\mathbb{P}^1_{\mathbb{C}}$ or X, prove that the tensor product of semistable vector bundles on Z is semistable.

13. For every integer h, show that X_h is a complete abstract curve.

Hint. Show that X_h is a Dedekind scheme by observing that the natural map $\pi_h : X_h \to X$ is finite étale. Define the degree map for X_h via the pushforward along π_h.

14. Let d, h, and r be integers with $h, r > 0$.

(1) Show that the vector bundle $\mathcal{O}_h(d, r)$ on X_h is semistable of rank r and degree d.

(2) If d and r are relatively prime, show that $\mathcal{O}_h(d, r)$ is stable.

15. For every closed interval $[a, b] \subseteq (0, 1)$ with $a \neq b$, show that the canonical continuous embedding $B_a^+ \hookrightarrow B_b^+$ is not surjective.

16. Given a p-adic field K, show that the principal ideal domain B_e is not Euclidean.

17. Assume that F is the tilt of $\mathbb{C}_K$ for a p-adic field K.

(1) Show that every vector bundle on X is isomorphic to $\mathcal{E}(D)$ for a unique isocrystal D over $\widehat{K^{\text{un}}}$ up to isomorphism.

(2) Find two isocrystals D' and D over $\widehat{K^{\text{un}}}$ with

$$\text{Hom}_{\text{Isoc}}(D', D) = 0 \quad \text{and} \quad \text{Hom}_{\mathcal{O}_X}(\mathcal{E}(D'), \mathcal{E}(D)) \neq 0.$$

18. Assume that F is the tilt of $\mathbb{C}_K$ for a p-adic field K and let $\widetilde{B}_{\text{cris}}$ denote either $B^+[1/t]$ or $B[1/t]$ for a cyclotomic uniformizer $t \in B_{\text{dR}}^+$.

(1) Show that $\widetilde{B}_{\text{cris}}$ is naturally a Γ_K-stable subring of B_{dR}.

(2) Show that $\widetilde{B}_{\text{cris}}$ is $(\mathbb{Q}_p, \Gamma_K)$-regular with an identification $\widetilde{B}_{\text{cris}}^{\Gamma_K} \cong K_0$.

Hint. Adapt the argument for Theorem 3.1.14 in Chapter III.

(3) For every p-adic Γ_K-representation V, show that $\widetilde{D}_{\text{cris}}(V) := (V \otimes_{\mathbb{Q}_p} \widetilde{B}_{\text{cris}})^{\Gamma_K}$ is naturally an isocrystal over K_0 with a canonical isomorphism

$$D_{\text{cris}}(V) \cong \widetilde{D}_{\text{cris}}(V).$$

Hint. Construct injective morphisms $D_{\text{cris}}(V) \hookrightarrow \widetilde{D}_{\text{cris}}(V)$ and $\widetilde{D}_{\text{cris}}(V) \hookrightarrow D_{\text{cris}}(V)$ after observing that Lemma 3.2.19 remains valid with $\widetilde{B}_{\text{cris}}$ in place of B_{cris}.

19. Assume that F is the tilt of $\mathbb{C}_K$ for a p-adic field K.

(1) Given a cyclotomic uniformizer $t \in B_{\text{dR}}^+$, prove that an element in $B^{\varphi=p}$ is invertible in $B[1/t]$ if and only if it is a $\mathbb{Q}_p^\times$-multiple of t.

(2) Prove that ∞ is the only closed point on X with finite Γ_K-orbit.

20. Assume that F is the tilt of $\mathbb{C}_K$ for a p-adic field K and let $\mathfrak{B}$ be a $(\mathbb{Q}_p, \Gamma_K)$-regular ring which contains B_e as a Γ_K-stable subring.

(1) Show that every Γ_K-equivariant vector bundle $\mathcal{V}$ on X naturally gives rise to a vector space $\mathcal{D}_{\mathfrak{B}}(\mathcal{V}) := (H^0(U, \mathcal{V}) \otimes_{B_e} \mathfrak{B})^{\Gamma_K}$ over $E := \mathfrak{B}^{\Gamma_K}$ with the following properties:

(i) There exists a canonical injective $\mathfrak{B}$-linear Γ_K-equivariant map

$$\alpha_{\mathcal{V}} : \mathcal{D}_{\mathfrak{B}}(\mathcal{V}) \otimes_E \mathfrak{B} \longrightarrow H^0(U, \mathcal{V}) \otimes_{B_e} \mathfrak{B}.$$

(ii) $\mathcal{D}_{\mathfrak{B}}(\mathcal{V})$ satisfies the inequality

$$\dim_E \mathcal{D}_{\mathfrak{B}}(\mathcal{V}) \leq \text{rk}(\mathcal{V})$$

with equality precisely when $\alpha_{\mathcal{V}}$ is an isomorphism.

(2) For every p-adic Γ_K-representation V, show that the $\mathcal{O}_X$-module $V \otimes_{\mathbb{Q}_p} \mathcal{O}_X$ is naturally a Γ_K-equivariant vector bundle on X with a canonical isomorphism

$$D_{\mathfrak{B}}(V) \cong \mathcal{D}_{\mathfrak{B}}(V \otimes_{\mathbb{Q}_p} \mathcal{O}_X).$$

CHAPTER V

Additional topics

1. Semistable representations

In this section, we define and study the semistable period ring and semistable representations. Our primary references for this section are the notes of Brinon-Conrad [**BC**, §9] and the notes of Fontaine-Ouyang [**FO**, §8].

1.1. The semistable period ring B_{st}

Throughout this section, we let K be a p-adic field with absolute Galois group Γ_K, inertia group I_K, and residue field k. We write $F := \mathbb{C}_K^\flat$ and denote the fraction field of $W(k)$ by K_0. In addition, we fix a cyclotomic uniformizer $t = \log(\varepsilon)$ of B_{dR}^+ for some $\varepsilon \in \mathbb{Z}_p(1)$ and a distinguished element $\xi = [p^\flat] - p \in A_{\mathrm{inf}}$ for some $p^\flat \in \mathcal{O}_F$ with $(p^\flat)^\sharp = p$.

Lemma 1.1.1. There exists a unique cocycle $\omega : \Gamma_K \to \mathbb{Z}_p(1)$ with

$$\gamma(p^\flat) = \varepsilon^{\omega(\gamma)} p^\flat \quad \text{for each } \gamma \in \Gamma_K.$$

Proof. The assertion is straightforward to verify. □

Definition 1.1.2. We refer to the cocycle $\omega : \Gamma_K \to \mathbb{Z}_p(1)$ given by Lemma 1.1.1 as the *logarithmic cocycle* associated to $p^\flat$.

Proposition 1.1.3. The tilted logarithm extends to a Γ_K-equivariant homomorphism $\log : F^\times \to B_{\mathrm{dR}}^+$ with an equality

$$\log(p^\flat) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{([p^\flat]/p - 1)^n}{n} = \sum_{n=1}^{\infty} \frac{\xi^n}{np^n}. \tag{1.1}$$

Proof. The tilted logarithm extends to a homomorphism $\mathcal{O}_F^\times \cong (1 + \mathfrak{m}_F) \times k_F^\times \to B_{\mathrm{dR}}^+$ with trivial image on $k_F^\times$, where $\mathfrak{m}_F$ and k_F respectively denote the maximal ideal and the residue field of $\mathcal{O}_F$, and consequently extends to a homomorphism $\log : F^\times \to B_{\mathrm{dR}}^+$ with the equality (1.1) as every $c \in F^\times$ admits an identity $c^m = (p^\flat)^n c'$ with $c' \in \mathcal{O}_F^\times$ for some $m, n \in \mathbb{Z}$. Hence we obtain the assertion as the tilted logarithm is Γ_K-equivariant by construction. □

Definition 1.1.4. We refer to the map log in Proposition 1.1.3 as the *extended tilted logarithm* and define the *semistable period ring* to be $B_{\mathrm{st}} := B_{\mathrm{cris}}[u]$ with $u := \log(p^\flat)$.

Remark. Let us explain Fontaine's insight behind the construction of B_{st}. Fontaine introduced the ring B_{st} to formulate an analogue of the crystalline comparison isomorphism for proper smooth varieties over K with semistable reduction. For an elliptic curve over K, having semistable reduction means that the mod p reduction may have a nodal singularity. A primordial example of an elliptic curve over K with semistable reduction is the *Tate curve* E_p which admits an identification $E_p(\overline{K}) \cong \overline{K}^\times/p^{\mathbb{Z}}$. After noting that $V_p(E_p)$ is isomorphic to the $\mathbb{Q}_p$-subspace of B_{dR} spanned by t and u, Fontaine constructed B_{st} as a $(\mathbb{Q}_p, \Gamma_K)$-regular ring containing B_{cris} such that $V_p(E_p)$ is B_{st}-admissible.

LEMMA 1.1.5. The group Γ_K acts on u via the equality

$$\gamma(u) = u + \omega(\gamma)t \quad \text{for each } \gamma \in \Gamma_K.$$

PROOF. The assertion is evident as log is Γ_K-equivariant by construction. □

Remark. Since Theorem 2.2.26 in Chapter III shows that Γ_K acts on t via the cyclotomic character, we deduce from Lemma 1.1.5 that the $\mathbb{Q}_p$-subspace of B_{dR} spanned by t and u is a nonsplit extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$.

PROPOSITION 1.1.6. The element $u \in B_{\text{dR}}^+$ is transcendental over the fraction field of B_{cris}.

PROOF. Suppose for contradiction that u is algebraic over the fraction field C_{cris} of B_{cris}. The element $u \in B_{\text{dR}}^+$ admits a unique minimal polynomial equation

$$u^d + c_1 u^{d-1} + \cdots + c_{d-1}u + c_d = 0 \quad \text{with } c_i \in C_{\text{cris}}.$$

For each $\gamma \in \Gamma_K$, we apply Lemma 1.1.5 to obtain an equality

$$(u + \omega(\gamma)t)^d + \gamma(c_1)(u + \omega(\gamma)t)^{d-1} + \cdots + \gamma(c_{d-1})(u + \omega(\gamma)t) + \gamma(c_d) = 0$$

and in turn find $c_1 = \gamma(c_1) + d\omega(\gamma)t$ by the uniqueness of the minimal polynomial equation. We see that $c_1 + du$ lies in $B_{\text{cris}}^{\Gamma_K} \cong K_0$ by Theorem 3.1.14 in Chapter III and consequently deduce that u is an element in C_{cris}.

Since C_{cris} naturally embeds into the fraction field of the p-adically complete ring $A_{\text{inf}}[[\xi/p]]$ by Proposition 3.1.3 in Chapter III, there exists an element $b \in A_{\text{inf}}[[\xi/p]]$ with $b \notin pA_{\text{inf}}[[\xi/p]]$ and $p^e bu \in A_{\text{inf}}[[\xi/p]]$ for some integer $e > 0$. Let us write $b = \sum a_i(\xi/p)^i$ with $a_i \in A_{\text{inf}}$. If each $\theta(a_i)$ is divisible by p in $\mathcal{O}_{\mathbb{C}_K}$, we see by Proposition 2.2.12 in Chapter III that each a_i lies in $pA_{\text{inf}} + \xi A_{\text{inf}}$ and in turn deduce that b is divisible by p in $A_{\text{inf}}[[\xi/p]]$, which is impossible. Now we take the smallest integer $m > 0$ with $\theta(a_m) \notin p\mathcal{O}_{\mathbb{C}_K}$ and obtain an identity

$$b = p\sum_{i<m} b_i \frac{\xi^i}{p^i} + b_m \frac{\xi^m}{p^m} + \sum_{i>m} b_i \frac{\xi^i}{p^i}$$

where each b_i is an element in A_{inf} with $\theta(b_m) \notin p\mathcal{O}_{\mathbb{C}_K}$. In addition, we fix an integer $n > e$ with $p^n > m$ and find

$$p^{n-1}u = \sum_{j<p^n} \frac{p^{n-1}}{j} \cdot \frac{\xi^j}{p^j} + \frac{1}{p} \cdot \frac{\xi^{p^n}}{p^{p^n}} + \sum_{j>p^n} \frac{p^{n-1}}{j} \cdot \frac{\xi^j}{p^j}.$$

Therefore we have an equality

$$\frac{b_m}{p} \cdot \frac{\xi^{m+p^n}}{p^{m+p^n}} = p^{n-1}ub - b\sum_{j<p^n} \frac{p^{n-1}}{j} \cdot \frac{\xi^j}{p^j} - b\sum_{j>p^n} \frac{p^{n-1}}{j} \cdot \frac{\xi^j}{p^j} - \sum_{i<m} b_i \frac{\xi^{i+p^n}}{p^{i+p^n}} - \sum_{i>m} \frac{b_i}{p} \cdot \frac{\xi^{i+p^n}}{p^{i+p^n}}.$$

It is not hard to see that the third term on the right side lies in $A_{\text{inf}}[[\xi/p]] + \xi^{m+p^n+1}B_{\text{dR}}^+$, while every other term on the right side belongs to either $A_{\text{inf}}[[\xi/p]]$ or $\xi^{m+p^n+1}B_{\text{dR}}^+$. Since the left side lies in $\xi^{m+p^n}B_{\text{dR}}^+$, we deduce that the right side represents a sum of elements in $(\xi/p)^{m+p^n}A_{\text{inf}}[[\xi/p]]$ and $\xi^{m+p^n+1}B_{\text{dR}}^+$. Hence we may write

$$\frac{b_m}{p} = b' + \xi b'' \quad \text{with } b' \in A_{\text{inf}}[[\xi/p]] \text{ and } b'' \in B_{\text{dR}}^+.$$

Now we find $\theta_{\text{dR}}^+(b_m/p) = \theta(b_m)/p \notin \mathcal{O}_{\mathbb{C}_K}$ and $\theta_{\text{dR}}^+(b' + \xi b'') = \theta_{\text{dR}}^+(b') \in \mathcal{O}_{\mathbb{C}_K}$, thereby obtaining a desired contradiction. □

Remark. Proposition 1.1.6 implies that B_{st} is isomorphic to the polynomial ring over B_{cris}.

Proposition 1.1.7. The ring B_{st} is naturally a filtered K_0-subalgebra of B_{dR} which is stable under the action of Γ_K.

Proof. We note that B_{cris} is canonically a K_0-subalgebra of B_{dR} by Proposition 3.1.8 in Chapter III and consequently find that B_{st} is naturally a filtered K_0-subalgebra of B_{dR} with $\mathrm{Fil}^n(B_{\mathrm{st}}) = B_{\mathrm{st}} \cap t^n B_{\mathrm{dR}}^+$ for every $n \in \mathbb{Z}$. Now we apply Lemma 1.1.5 to see that B_{st} is stable under the Γ_K-action, thereby establishing the desired assertion. □

Remark. It is worthwhile to mention that the embedding $B_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$ is not completely canonical; indeed, it depends on the definition of the extended tilted logarithm which involves some choices.

Proposition 1.1.8. The natural Γ_K-equivariant map $B_{\mathrm{st}} \otimes_{K_0} K \to B_{\mathrm{dR}}$ is injective.

Proof. Proposition 3.1.9 in Chapter III shows that the natural map $B_{\mathrm{cris}} \otimes_{K_0} K \to B_{\mathrm{dR}}$ is injective. Moreover, since K is finite over K_0 by Proposition 2.2.19 in Chapter III, we deduce from Proposition 1.1.6 that u is trascendental over the fraction field of $B_{\mathrm{cris}} \otimes_{K_0} K$. Hence we find that the kernel of the natural map $B_{\mathrm{st}} \otimes_{K_0} K \to B_{\mathrm{dR}}$ is zero as desired. □

Proposition 1.1.9. There exists a natural Γ_K-equivariant graded K-algebra isomorphism

$$\mathrm{gr}(B_{\mathrm{st}} \otimes_{K_0} K) \cong B_{\mathrm{HT}}.$$

Proof. Theorem 2.2.26 in Chapter III and Proposition 1.1.8 together imply that the canonical filtered K-algebra homomorphism $B_{\mathrm{st}} \otimes_{K_0} K \to B_{\mathrm{dR}}$ yields an injective graded K-algebra homomorphism

$$\mathrm{gr}(B_{\mathrm{st}} \otimes_{K_0} K) \longrightarrow \mathrm{gr}(B_{\mathrm{dR}}) \cong B_{\mathrm{HT}}. \tag{1.2}$$

This map is surjective as it restricts to an isomorphism $\mathrm{gr}(B_{\mathrm{cris}} \otimes_{K_0} K) \cong B_{\mathrm{HT}}$ given by Proposition 3.1.10 in Chapter III. Moreover, since each $\mathrm{Fil}^n(B_{\mathrm{st}}) = B_{\mathrm{st}} \cap t^n B_{\mathrm{dR}}^+$ is stable under the Γ_K-action by Theorem 2.2.26 in Chapter III, we obtain a canonical action of Γ_K on $\mathrm{gr}(B_{\mathrm{st}} \otimes_{K_0} K)$ and in turn deduce that the map (1.2) is Γ_K-equivariant. □

Theorem 1.1.10 (Fontaine [**Fon94a**]). The ring B_{st} is $(\mathbb{Q}_p, \Gamma_K)$-regular with $B_{\mathrm{st}}^{\Gamma_K} \cong K_0$.

Proof. The ring B_{st} is a subring of the field B_{dR} and thus is an integral domain. Proposition 1.1.7 implies that the fraction field C_{st} of B_{st} is a K_0-subalgebra of B_{dR} which is stable under the Γ_K-action. In addition, Theorem 2.2.26 in Chapter III and Proposition 1.1.8 together yield natural injective K-algebra homomorphisms

$$B_{\mathrm{st}}^{\Gamma_K} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}^{\Gamma_K} \cong K \quad \text{and} \quad C_{\mathrm{st}}^{\Gamma_K} \otimes_{K_0} K \hookrightarrow B_{\mathrm{dR}}^{\Gamma_K} \cong K.$$

Therefore we have $K_0 \cong B_{\mathrm{st}}^{\Gamma_K} \cong C_{\mathrm{st}}^{\Gamma_K}$.

It remains to prove that every nonzero $b \in B_{\mathrm{cris}}$ with $\mathbb{Q}_p b$ being stable under the Γ_K-action is a unit. We apply Proposition 2.2.23 to write $b = t^n b'$ for some $b' \in (B_{\mathrm{dR}}^+)^\times$ and $n \in \mathbb{Z}$. We observe that t is a unit in B_{st} and in turn find $b' = bt^{-n} \in B_{\mathrm{st}}$. Moreover, Theorem 2.2.26 in Chapter III implies that $\mathbb{Q}_p b'$ is stable under the Γ_K-action. Hence we may replace b by b' to assume that b lies in $(B_{\mathrm{dR}}^+)^\times$. Proposition 3.1.13 in Chapter III yields a polynomial equation

$$b^d + c_1 b^{d-1} + \cdots + c_{d-1} b + c_d = 0 \quad \text{with } c_d \neq 0$$

where each c_i is an element in the fraction field $\widehat{K_0^{\mathrm{un}}}$ of $W(\overline{k})$. Now we find

$$b^{-1} = -c_d^{-1}(b^{d-1} + c_1 b^{d-2} + \cdots + c_{d-1}) \in B_{\mathrm{cris}}$$

by noting that $\widehat{K_0^{\mathrm{un}}}$ naturally embeds into B_{st}, thereby completing the proof. □

PROPOSITION 1.1.11. The Frobenius endomorphism of B_{cris} canonically extends to a Γ_K-equivariant endomorphism φ on B_{st} with $\varphi(u) = pu$.

PROOF. The assertion is evident by Lemma 1.1.5 and Proposition 1.1.6. □

Remark. The equality $\varphi(u) = pu$ ensures that φ is compatible with the Frobenius automorphism on $F^\times$ via the extended tilted logarithm.

Definition 1.1.12. We refer to the map φ given by Proposition 1.1.11 as the *Frobenius endomorphism* of B_{st}.

PROPOSITION 1.1.13. The Frobenius endomorphism of B_{st} is injective.

PROOF. Since the Frobenius endomorphism of B_{cris} is injective by Theorem 1.1.13 in Chapter III, we deduce the desired assertion from Proposition 1.1.6. □

PROPOSITION 1.1.14. There exists a unique B_{cris}-linear map $N : B_{\text{st}} \to B_{\text{st}}$ such that each element $b = \sum b_n u^n \in B_{\text{st}}$ with $b_n \in B_{\text{cris}}$ satisfies the equality

$$N(b) = -\sum_{n \geq 1} n b_n u^{n-1}.$$

PROOF. The assertion is an immediate consequence of Proposition 1.1.6. □

Remark. We may identify N as the unique B_{cris}-derivation on B_{st} which maps u to -1.

Definition 1.1.15. We refer to the map N given by Proposition 1.1.14 as the *monodromy operator* on B_{st}.

Remark. Let us provide some motivation for the construction of N. Fix a prime $\ell \neq p$ and take the cocycle $\psi : I_K \to \mathbb{Z}_\ell(1)$ given by the I_K-action on the ℓ-power roots of a uniformizer. For every proper smooth K-variety X with semistable reduction, the ℓ-adic Γ_K-representation $V := H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_\ell)$ turns out to admit a nilpotent endomorphism N_V such that each $\gamma \in I_K$ acts via $\exp(\psi(\gamma)N_V)$. The endomorphism N_V serves as an analogue of the monodromy operator from complex geometry. Moreover, the identification of each $\gamma \in I_K$ with $\exp(\psi(\gamma)N_V)$ indicates that N_V essentially behaves as the derivative of a representation does. Fontaine introduced the monodromy operator N on B_{st} to provide an analogue of the monodromy operator for p-adic Γ_K-representations.

PROPOSITION 1.1.16. The monodromy operator N on B_{st} satisfies the following properties:

(i) N is surjective with $\ker(N) \cong B_{\text{cris}}$.
(ii) N is Γ_K-equivariant and admits the identity $N \circ \varphi = p\varphi \circ N$.

PROOF. Property (i) is straightforward to verify by Proposition 1.1.6. Moreover, given an integer $n \geq 1$ we find by Lemma 1.1.5 that each $\gamma \in \Gamma_K$ satisfies the relation

$$N(\gamma(u^n)) = N((u + \omega(\gamma)t)^n) = -n(u + \omega(\gamma)t)^{n-1} = -\gamma(nu^{n-1}) = \gamma(N(u^n))$$

and also see that φ yields the equality

$$N(\varphi(u^n)) = N(p^n u^n) = -np^n u^{n-1} = -p\varphi(nu^{n-1}) = p\varphi(N(u^n)),$$

thereby establishing property (ii). □

1.2. Properties of semistable representations

For the rest of this section, we denote by σ the Frobenius automorphism of K_0 and by Vect_{K_0} the category of K_0-vector spaces.

Definition 1.2.1. For a K_0-vector space V, we write $V_K = V \otimes_{K_0} K$.

(1) A (φ, N)*-module* over K_0 is an isocrystal D over K_0 together with a K_0-linear endomorphism N_D, called the *monodromy operator* of D, which satisfies the equality

$$N_D \circ \varphi_D = p\varphi_D \circ N_D.$$

(2) A *filtered* (φ, N)*-module* over K is a (φ, N)-module D over K_0 such that D_K is a filtered K-vector space.

(3) A K_0-linear map $f : D \to D'$ for (φ, N)-modules D and D' over K_0 is a *morphism of* (φ, N)*-modules* if it satisfies the identities

$$f \circ \varphi_D = \varphi_{D'} \circ f \quad \text{ and } \quad f \circ N_D = N_{D'} \circ f.$$

(4) Given two filtered (φ, N)-modules D and D' over K, a morphism $f : D \to D'$ of (φ, N)-modules is K*-filtered* if the induced map $f_K : D_K \to D'_K$ is filtered.

Remark. A K-filtered isomorphism of (φ, N)-modules is a bijective K-filtered morphism of (φ, N)-modules with a K-filtered inverse.

Proposition 1.2.2. Let D be a filtered (φ, N)-module over K.

(1) Given a filtered (φ, N)-module D' over K, the tensor product $D \otimes_{K_0} D'$ is naturally a filtered (φ, N)-module over K with monodromy operator $N_D \otimes 1 + 1 \otimes N_{D'}$.

(2) The dual $D^\vee = \mathrm{Hom}_{K_0}(D, K_0)$ is naturally a filtered (φ, N)-module over K with monodromy operator given by the dual map of $-N_D$.

Proof. The assertions follow from Proposition 3.2.4 in Chapter III. □

Remark. We note that the formulas for the monodromy operators in Proposition 1.2.2 are analogous to the formulas for the tensor products and duals of Lie algebra representations.

Proposition 1.2.3. Let V be a p-adic Γ_K-representation.

(1) The K_0-vector space $D_{\mathrm{st}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{\Gamma_K}$ is naturally a filtered (φ, N)-module over K with Frobenius automorphism $1 \otimes \varphi$, monodromy operator $1 \otimes N$, and

$$\mathrm{Fil}^n(D_{\mathrm{st}}(V)_K) = (V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^n(B_{\mathrm{st}} \otimes_{K_0} K))^{\Gamma_K} \quad \text{for each } n \in \mathbb{Z}. \tag{1.3}$$

(2) There exists a canonical K-filtered isomorphism of isocrystals

$$D_{\mathrm{cris}}(V) \cong D_{\mathrm{st}}(V)^{1 \otimes N = 0}.$$

Proof. Theorem 1.2.1 in Chapter III and Theorem 1.1.10 together imply that $D_{\mathrm{st}}(V)$ is a finite dimensional K_0-vector space. In addition, we find

$$D_{\mathrm{st}}(V)_K = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{\Gamma_K} \otimes_{K_0} K = (V \otimes_{\mathbb{Q}_p} (B_{\mathrm{st}} \otimes_{K_0} K))^{\Gamma_K}$$

and in turn deduce from Proposition 1.1.8 that $D_{\mathrm{st}}(V)_K$ is a filtered K-vector space with the identification (1.3). Meanwhile, since $1 \otimes \varphi$ is σ-semilinear by the fact that φ extends σ, it is bijective on $D_{\mathrm{cris}}(V)$ by Lemma 3.2.5 in Chapter III and Proposition 1.1.13. Now we apply Proposition 1.1.16 to find

$$(1 \otimes N) \circ (1 \otimes \varphi) = p(1 \otimes \varphi) \circ (1 \otimes N)$$

and thus obtain statement (1). Statement (2) is straightforward to verify by statement (1). □

Definition 1.2.4. Let V be a p-adic Γ_K-representation.

(1) We refer to $D_{\mathrm{st}}(V)$ in Proposition 1.2.3 as the *filtered* (φ, N)*-module associated to* V.
(2) We say that V is *semistable* if it is B_{st}-admissible.

PROPOSITION 1.2.5. A p-adic Γ_K-representation V is crystalline if and only if it is semistable with trivial monodromy operator on $D_{\mathrm{st}}(V)$.

PROOF. By Proposition 1.2.3, we may identify $D_{\mathrm{cris}}(V)$ with the kernel of the monodromy operator on $D_{\mathrm{st}}(V)$. If V is crystalline, we apply Theorem 1.2.1 in Chapter III to find

$$\dim_{\mathbb{Q}_p} V = \dim_{K_0} D_{\mathrm{cris}}(V) \leq \dim_{K_0} D_{\mathrm{st}}(V) \leq \dim_{\mathbb{Q}_p} V$$

and in turn deduce that V is semistable with trivial monodromy operator on $D_{\mathrm{st}}(V)$. Convsersely, if V is semistable with trivial monodromy operator on $D_{\mathrm{st}}(V)$, we obtain a canonical isomorphism $D_{\mathrm{cris}}(V) \cong D_{\mathrm{st}}(V)$ and thus see that V is crystalline. □

Example 1.2.6. Theorem 2.2.26 in Chapter III and Lemma 1.1.5 together imply that the $\mathbb{Q}_p$-subspace V of B_{st} spanned by t and u is a p-adic Γ_K-representation. Since $D_{\mathrm{st}}(V)$ contains K_0-linearly independent elements $t \otimes t^{-1}$ and $-t \otimes ut^{-1} + u \otimes 1$, we find that the inequality

$$\dim_{K_0} D_{\mathrm{st}}(V) \leq \dim_{\mathbb{Q}_p} V = 2$$

given by Theorem 1.2.1 in Chapter III must be an equality, which means that V is semistable. In addition, we observe that $-t \otimes ut^{-1} + u \otimes 1$ does not map to 0 under $1 \otimes N$ and in turn deduce from Proposition 1.2.5 that V is not crystalline.

Remark. As mentioned after Definition 1.1.4, the p-adic Γ_K-representation V is isomorphic to the rational Tate module of the Tate curve E_p. Hence its dual $H^1_{\text{ét}}(E_p, \mathbb{Q}_p)$ is semistable by Proposition 1.2.7 in Chapter III and Example 1.2.6. In fact, the work of Tsuji [**Tsu99**] shows that the p-adic étale cohomology of every proper smooth K-variety with semistable reduction is semistable, as originally conjectured by Fontaine [**Fon94b**].

LEMMA 1.2.7. If a (φ, N)-module D over K_0 has rank 1, its monodromy operator vanishes.

PROOF. Take a K_0-basis element e for D. We may write $\varphi_D(e) = ce$ and $N_D(e) = c'e$ for some $c, c' \in K_0$. By the relation $N_D \circ \varphi_D = p\varphi_D \circ N_D$, we obtain the equality $cc' = pc\sigma(c')$. Since we have $c \neq 0$ and $K_0^{\sigma = p^{-1}} = 0$, we find $c' = 0$ and in turn see that N_D vanishes. □

Remark. In fact, the monodromy operator of an arbitrary (φ, N)-module is nilpotent.

PROPOSITION 1.2.8. A p-adic Γ_K-representation V of dimension 1 is semistable if and only if it is crystalline.

PROOF. The assertion is evident by Proposition 1.2.5 and Lemma 1.2.7. □

Example 1.2.9. Example 3.2.10 in Chapter III and Proposition 1.2.8 together show that every Tate twist $\mathbb{Q}_p(n)$ of $\mathbb{Q}_p$ is semistable; indeed, $D_{\mathrm{st}}(\mathbb{Q}_p(n))$ is naturally isomorphic to the simple isocrystal of slope $-n$ with trivial monodromy operator and admits identifications

$$\mathrm{Fil}^m(D_{\mathrm{st}}(\mathbb{Q}_p(n))_K) \cong \begin{cases} K & \text{for } m \leq -n, \\ 0 & \text{for } m > -n. \end{cases}$$

LEMMA 1.2.10. Given an integer n, a p-adic Γ_K-representation V is semistable if and only if its Tate twist $V(n)$ is semistable.

PROOF. Since we have $V(n) \cong V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(n)$ and $V \cong V(n) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(-n)$, the assertion follows from Proposition 1.2.4 in Chapter III and Example 1.2.9. □

Proposition 1.2.11. If a p-adic Γ_K-representation V is semistable, it is de Rham with a natural K-linear filtered isomorphism

$$D_{\mathrm{st}}(V)_K \cong D_{\mathrm{dR}}(V).$$

Proof. Proposition 1.1.7 and Proposition 1.1.8 together show that $B_{\mathrm{st}} \otimes_{K_0} K$ is naturally a filtered K-subalgebra of B_{dR} with

$$\mathrm{Fil}^n(B_{\mathrm{st}} \otimes_{K_0} K) = (B_{\mathrm{st}} \otimes_{K_0} K) \cap \mathrm{Fil}^n(B_{\mathrm{dR}}) \quad \text{for every } n \in \mathbb{Z}.$$

Therefore Proposition 1.2.3 yields a natural injective K-linear filtered map

$$D_{\mathrm{st}}(V)_K = (V \otimes_{\mathbb{Q}_p} (B_{\mathrm{st}} \otimes_{K_0} K))^{\Gamma_K} \hookrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{\Gamma_K} = D_{\mathrm{dR}}(V)$$

with an identification

$$\mathrm{Fil}^n(D_{\mathrm{st}}(V)_K) = D_{\mathrm{st}}(V)_K \cap \mathrm{Fil}^n(D_{\mathrm{dR}}(V)) \quad \text{for every } n \in \mathbb{Z}.$$

In addition, we find

$$\dim_{K_0} D_{\mathrm{st}}(V) = \dim_K D_{\mathrm{st}}(V)_K \leq \dim_K D_{\mathrm{dR}}(V) \leq \dim_{\mathbb{Q}_p} V$$

where the last inequality follows from Theorem 1.2.1 in Chapter III. Since V is semistable, we see that both inequalities should be equalities and in turn establish the desired assertion. □

Example 1.2.12. Given a continuous character $\eta : \Gamma_K \to \mathbb{Q}_p^\times$ with $\eta(I_K)$ being nontrivially finite, Proposition 3.2.27 in Chapter III and Proposition 1.2.8 together imply that $\mathbb{Q}_p(\eta)$ is de Rham but not semistable.

Proposition 1.2.13. Let V be a p-adic Γ_K-representation and L be a finite unramified extension of K with residue field l. Denote by L_0 the fraction field of $W(l)$.

(1) There exists an L-filtered isomorphism of isocrystals

$$D_{\mathrm{st},K}(V) \otimes_{K_0} L_0 \cong D_{\mathrm{st},L}(V)$$

where we set $D_{\mathrm{st},K}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{\Gamma_K}$ and $D_{\mathrm{st},L}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{\Gamma_L}$.

(2) V is semistable if and only if it is semistable as a Γ_L-representation.

Proof. Lemma 2.4.16 in Chapter III shows that L is a p-adic field. Moreover, L and L_0 are respectively Galois over K and K_0 with natural isomorphisms

$$\mathrm{Gal}(L/K) \cong \mathrm{Gal}(L_0/K_0) \cong \mathrm{Gal}(l/k).$$

Hence we find

$$D_{\mathrm{st},K}(V) = D_{\mathrm{st},L}(V)^{\mathrm{Gal}(L/K)} = D_{\mathrm{st},L}(V)^{\mathrm{Gal}(L_0/K_0)}$$

and in turn apply Lemma 2.4.16 in Chapter III to obtain a natural bijective L_0-linear map

$$D_{\mathrm{st},K}(V) \otimes_{K_0} L_0 \longrightarrow D_{\mathrm{st},L}(V). \tag{1.4}$$

This map is evidently a morphism of (φ, N)-modules. In addition, by Proposition 2.4.17 in Chapter III and Proposition 1.2.11, the map (1.4) induces an L-linear filtered isomorphism

$$(D_{\mathrm{st},K}(V) \otimes_{K_0} K) \otimes_K L \cong D_{\mathrm{st},L}(V) \otimes_{L_0} L.$$

We deduce that the map (1.4) is an L-filtered isomorphism of (φ, N)-modules and consequently establish statement (1). Statement (2) is an immediate consequence of statement (1). □

Remark. We can show that Proposition 1.2.13 remains valid for $L = \widehat{K^{\mathrm{un}}}$ by the remark following Lemma 2.4.16 in Chapter III. On the other hand, Example 1.2.12 implies that Proposition 1.2.13 fails for a ramified extension L of K.

For the rest of this section, we denote by $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(\Gamma_K)$ the category of semistable Γ_K-representations and by $\mathrm{MF}_K^{\varphi,N}$ the category of filtered (φ, N)-module over K.

PROPOSITION 1.2.14. Every $V \in \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(\Gamma_K)$ admits a natural Γ_K-equivariant isomorphism

$$D_{\mathrm{st}}(V) \otimes_{K_0} B_{\mathrm{st}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$$

with the following properties:

(i) It is compatible with the Frobenius endomorphisms and the monodromy operators.

(ii) It induces a filtered isomorphism of vector spaces

$$D_{\mathrm{st}}(V)_K \otimes_K (B_{\mathrm{st}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{st}} \otimes_{K_0} K).$$

PROOF. Theorem 1.2.1 in Chapter III implies that the natural B_{st}-linear map

$$D_{\mathrm{st}}(V) \otimes_{K_0} B_{\mathrm{st}} \longrightarrow (V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}) \otimes_{K_0} B_{\mathrm{st}} \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{st}} \otimes_{K_0} B_{\mathrm{st}}) \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$$

is Γ_K-equivariant and bijective. Moreover, this map is compatible with the natural Frobenius endomorphisms and monodromy actions on $D_{\mathrm{st}}(V) \otimes_{K_0} B_{\mathrm{st}}$ and $V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$. Let us now consider the induced K-linear bijective map

$$D_{\mathrm{st}}(V)_K \otimes_K (B_{\mathrm{st}} \otimes_{K_0} K) \longrightarrow V \otimes_{\mathbb{Q}_p} (B_{\mathrm{st}} \otimes_{K_0} K).$$

It is straightforward to verify that this map is filtered. Therefore by Proposition 2.3.10 in Chapter III, it suffices to prove the bijectivity of the graded map

$$\mathrm{gr}\left(D_{\mathrm{st}}(V)_K \otimes_K (B_{\mathrm{st}} \otimes_{K_0} K)\right) \longrightarrow \mathrm{gr}\left(V \otimes_{\mathbb{Q}_p} (B_{\mathrm{st}} \otimes_{K_0} K)\right). \tag{1.5}$$

Proposition 2.4.3 in Chapter III and Proposition 1.2.11 show that V is Hodge-Tate with

$$\mathrm{gr}(D_{\mathrm{st}}(V)_K) \cong \mathrm{gr}(D_{\mathrm{dR}}(V)) \cong D_{\mathrm{HT}}(V)$$

We apply Proposition 2.3.9 in Chapter III and Proposition 3.1.10 to obtain isomorphisms

$$\mathrm{gr}\left(D_{\mathrm{st}}(V)_K \otimes_K (B_{\mathrm{st}} \otimes_{K_0} K)\right) \cong \mathrm{gr}(D_{\mathrm{st}}(V)_K) \otimes_K \mathrm{gr}(B_{\mathrm{st}} \otimes_{K_0} K) \cong D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}},$$
$$\mathrm{gr}\left(V \otimes_{\mathbb{Q}_p} (B_{\mathrm{st}} \otimes_{K_0} K)\right) \cong V \otimes_{\mathbb{Q}_p} \mathrm{gr}(B_{\mathrm{st}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}.$$

Now we identify the map (1.5) with the natural isomorphism

$$D_{\mathrm{HT}}(V) \otimes_K B_{\mathrm{HT}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{HT}}$$

given by Theorem 1.2.1 in Chapter III and thus establish the desired assertion. □

PROPOSITION 1.2.15. The functor D_{st} with values in $\mathrm{MF}_K^{\varphi,N}$ is faithful and exact on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(\Gamma_K)$.

PROOF. Since the forgetful functor $\mathrm{MF}_K^{\varphi,N} \to \mathrm{Vect}_{K_0}$ is faithful, Proposition 1.2.2 in Chapter III implies that D_{st} is faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(\Gamma_K)$. Hence it remains to verify that D_{st} is exact on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(\Gamma_K)$. Consider an exact sequence of semistable Γ_K-representations

$$0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$$

By Proposition 1.2.2 in Chapter III, this sequence yields an exact sequence of (φ, N)-modules

$$0 \longrightarrow D_{\mathrm{st}}(U) \longrightarrow D_{\mathrm{st}}(V) \longrightarrow D_{\mathrm{st}}(W) \longrightarrow 0. \tag{1.6}$$

Moreover, we use Proposition 1.2.11 to identify the induced sequence of filtered vector spaces

$$0 \longrightarrow D_{\mathrm{st}}(U)_K \longrightarrow D_{\mathrm{st}}(V)_K \longrightarrow D_{\mathrm{st}}(W)_K \longrightarrow 0$$

with the exact sequence of filtered vector spaces

$$0 \longrightarrow D_{\mathrm{dR}}(U) \longrightarrow D_{\mathrm{dR}}(V) \longrightarrow D_{\mathrm{dR}}(W) \longrightarrow 0$$

given by Proposition 2.4.9 in Chapter III. Therefore the sequence (3.6) is exact in $\mathrm{MF}_K^{\varphi,N}$. □

Proposition 1.2.16. Given a semistable Γ_K-representation V, every subquotient W of V is semistable with $D_{\mathrm{st}}(W)$ naturally identified as a subquotient of $D_{\mathrm{st}}(V)$ in $\mathrm{MF}_K^{\varphi,N}$.

Proof. The assertion is an immediate consequence of Proposition 1.2.3 in Chapter III and Proposition 1.2.15. □

Proposition 1.2.17. Given two semistable Γ_K-representations V and W, their tensor product $V \otimes_{\mathbb{Q}_p} W$ is semistable with a natural K-filtered isomorphism of (φ, N)-modules

$$D_{\mathrm{st}}(V) \otimes_{K_0} D_{\mathrm{st}}(W) \cong D_{\mathrm{st}}(V \otimes_{\mathbb{Q}_p} W). \tag{1.7}$$

Proof. Proposition 1.2.4 in Chapter III shows that $V \otimes_{\mathbb{Q}_p} W$ is semistable and yields the desired isomorphism (1.7) as a K_0-linear bijection. It is straightforward to verify that the map (1.7) is a K-filtered morphism of (φ, N)-modules. Moreover, we use Proposition 1.2.11 to identify the induced map

$$D_{\mathrm{st}}(V)_K \otimes_K D_{\mathrm{st}}(W)_K \longrightarrow D_{\mathrm{st}}(V \otimes_{\mathbb{Q}_p} W)_K.$$

with the natural K-linear filtered isomorphism

$$D_{\mathrm{dR}}(V) \otimes_K D_{\mathrm{dR}}(W)_K \cong D_{\mathrm{dR}}(V \otimes_{\mathbb{Q}_p} W)$$

given by Proposition 2.4.11 in Chapter III. Hence we deduce that the map (1.7) is a K-filtered isomorphism of (φ, N)-modules, thereby completing the proof. □

Proposition 1.2.18. Given a semistable Γ_K-representation V and a positive integer n, both $\wedge^n(V)$ and $\mathrm{Sym}^n(V)$ are semistable with natural K-filtered isomorphisms of (φ, N)-modules

$$\wedge^n(D_{\mathrm{st}}(V)) \cong D_{\mathrm{st}}(\wedge^n(V)) \quad \text{and} \quad \mathrm{Sym}^n(D_{\mathrm{st}}(V)) \cong D_{\mathrm{st}}(\mathrm{Sym}^n(V)).$$

Proof. Proposition 1.2.5 in Chapter III shows that $\wedge^n(V)$ and $\mathrm{Sym}^n(V)$ are semistable. Moreover, Proposition 1.2.5 in Chapter III yields the desired isomorphisms as K_0-linear bijections. Proposition 1.2.16 and Proposition 1.2.17 imply that these maps are K-filtered isomorphisms of (φ, N)-modules. □

Example 1.2.19. Given a semistable Γ_K-representation V, we have

$$\mu(D_{\mathrm{st}}(V(n))) = \mu(D_{\mathrm{st}}(V)) - n \quad \text{for each } n \in \mathbb{Z}$$

by Example 1.2.9, Proposition 1.2.17, and Proposition 1.2.18.

Proposition 1.2.20. For every semistable Γ_K-representation V, the dual representation $V^\vee$ is semistable with a natural K-filtered perfect pairing of (φ, N)-modules

$$D_{\mathrm{st}}(V) \otimes_{K_0} D_{\mathrm{st}}(V^\vee) \cong D_{\mathrm{st}}(V \otimes_{\mathbb{Q}_p} V^\vee) \longrightarrow D_{\mathrm{st}}(\mathbb{Q}_p).$$

Proof. Proposition 1.2.7 in Chapter III shows that $V^\vee$ is semistable and yields the desired pairing as a K_0-linear perfect pairing. This pairing is a K-filtered morphism of (φ, N)-modules over K_0 by Proposition 1.2.17 and thus gives rise to a K-filtered bijective morphism of (φ, N)-modules

$$D_{\mathrm{st}}(V)^\vee \longrightarrow D_{\mathrm{st}}(V^\vee). \tag{1.8}$$

Moreover, we apply Proposition 1.2.11 to identify the induced K-linear filtered map

$$D_{\mathrm{st}}(V)_K^\vee \longrightarrow D_{\mathrm{st}}(V^\vee)_K$$

with the natural K-linear filtered isomorphism

$$D_{\mathrm{dR}}(V) \cong D_{\mathrm{dR}}(V^\vee)$$

given by Proposition 2.4.14 in Chapter III. Now we deduce that the map (1.8) is a K-filtered isomorphism of (φ, N)-modules, thereby completing the proof. □

For the rest of this section, we generally write φ and N respectively for maps naturally induced by the Frobenius endomorphism and the monodromy operator on B_{st}.

Proposition 1.2.21. Every semistable Γ_K-representation V admits canonical isomorphisms

$$\begin{aligned} V &\cong (D_{\mathrm{st}}(V) \otimes_{K_0} B_{\mathrm{st}})^{\varphi=1, N=0} \cap \mathrm{Fil}^0 \left(D_{\mathrm{st}}(V)_K \otimes_K (B_{\mathrm{st}} \otimes_{K_0} K)\right) \\ &\cong (D_{\mathrm{st}}(V) \otimes_{K_0} B_{\mathrm{st}})^{\varphi=1, N=0} \cap \mathrm{Fil}^0 \left(D_{\mathrm{st}}(V)_K \otimes_K B_{\mathrm{dR}}\right). \end{aligned}$$

Proof. Proposition 1.2.14 yields a natural Γ_K-equivariant isomorphism

$$D_{\mathrm{st}}(V) \otimes_{K_0} B_{\mathrm{st}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}$$

which is compatible with the Frobenius endomorphisms and the monodromy operators. Moreover, this isomorphism gives rise to a canonical filtered isomorphism

$$D_{\mathrm{st}}(V)_K \otimes_K (B_{\mathrm{st}} \otimes_{K_0} K) \cong V \otimes_{\mathbb{Q}_p} (B_{\mathrm{st}} \otimes_{K_0} K),$$

which in turn yields a natural filtered isomorphism

$$D_{\mathrm{st}}(V)_K \otimes_K B_{\mathrm{dR}} \cong D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}.$$

Therefore we obtain canonical isomorphisms

$$\begin{aligned} (D_{\mathrm{st}}(V) \otimes_{K_0} B_{\mathrm{st}})^{\varphi=1, N=0} &\cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}^{\varphi=1, N=0}, \\ \mathrm{Fil}^0 \left(D_{\mathrm{st}}(V)_K \otimes_K (B_{\mathrm{st}} \otimes_{K_0} K)\right) &\cong V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^0(B_{\mathrm{st}} \otimes_{K_0} K), \\ \mathrm{Fil}^0 \left(D_{\mathrm{st}}(V)_K \otimes_K B_{\mathrm{dR}}\right) &\cong V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}^{+}. \end{aligned}$$

Since we have an identification $B_e \cong B_{\mathrm{st}}^{\varphi=1, N=0}$ given by Proposition 1.1.16, we establish the desired assertion by Lemma 3.2.20 in Chapter III. □

Theorem 1.2.22 (Fontaine [**Fon94b**]). The functor D_{st} with values in $\mathrm{MF}_K^{\varphi, N}$ is exact and fully faithful on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(\Gamma_K)$.

Proof. By Proposition 1.2.15, we only need to prove that D_{st} is full on $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{st}}(\Gamma_K)$. Let V and W be arbitrary semistable Γ_K-representations. Consider an arbitrary morphism $f : D_{\mathrm{st}}(V) \to D_{\mathrm{st}}(W)$ in $\mathrm{MF}_K^{\varphi, N}$. Proposition 1.2.14 yields a Γ_K-equivariant B_{st}-linear map

$$V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}} \cong D_{\mathrm{st}}(V) \otimes_{K_0} B_{\mathrm{st}} \xrightarrow{f \otimes 1} D_{\mathrm{st}}(W) \otimes_{K_0} B_{\mathrm{st}} \cong W \otimes_{\mathbb{Q}_p} B_{\mathrm{st}}.$$

Moreover, Proposition 1.2.21 implies that this map restricts to a $\mathbb{Q}_p$-linear map $\phi : V \to W$. Now we identify f with the restriction of $\phi \otimes 1$ on $(V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{\Gamma_K}$ under the identification

$$(V \otimes_{\mathbb{Q}_p} B_{\mathrm{st}})^{\Gamma_K} \cong (D_{\mathrm{st}}(V) \otimes_{K_0} B_{\mathrm{st}})^{\Gamma_K} \cong D_{\mathrm{st}}(V)$$

and in turn deduce that f corresponds to ϕ under the functor D_{st}. □

Definition 1.2.23. Let D be a filtered (φ, N)-module over K.

(1) We say that D is *weakly admissible* if every nonzero filtered (φ, N)-submodule D' of D satisfies the inequality $\deg^{\bullet}(D') \leq \deg(D')$ with equality for $D' = D$.
(2) We say that D is *admissible* if it admits an isomorphism $D \simeq D_{\mathrm{st}}(V)$ for some semistable Γ_K-reprsentation V.

Remark. While every weakly admissible filtered isocrystal is naturally a weakly admissible filtered (φ, N)-module with zero monodromy operator, there exist weakly admissible filtered (φ, N)-modules over K which are not weakly admissible filtered isocrystals. Nonetheless, most statements about weakly admissible filtered isocrystals proved in Chapter III have generalizations for weakly admissible filtered (φ, N)-modules.

We close this section by stating two important theorems on semistable Γ_K-representations without providing proofs.

THEOREM 1.2.24 (Colmez-Fontaine [**CF00**]). A filtered (φ, N)-module over K is admissible if and only if it is weakly admissible.

Remark. In fact, Theorem 1.2.24 is not difficult to prove by adapting our argument for Theorem 3.2.22 in Chapter IV. Here we list some key analogues of statements proved in Chapter IV about isocrystals and their associated vector bundles on the Fargues-Fontaine curve.

(1) Every filtered (φ, N)-module D over K naturally gives rise to a Γ_K-equivariant vector bundle $\mathcal{F}_{\text{st}}(D)$ on X with canonical isomorphisms
$$H^0(U, \mathcal{F}_{\text{st}}(D)) \cong (D \otimes_{K_0} B_{\text{st}})^{\varphi=1,N=0} \quad \text{and} \quad \widehat{\mathcal{F}_{\text{st}}(D)_\infty} \cong \text{Fil}^0(D_K \otimes_K B_{\text{dR}}),$$
where $\widehat{\mathcal{F}_{\text{st}}(D)_\infty}$ denotes the completed stalk of $\mathcal{F}_{\text{st}}(D)$ at ∞.

(2) Every Γ_K-equivariant vector bundle $\mathcal{V}$ on X naturally gives rise to a (φ, N)-module $D_{\text{st}}(\mathcal{V}) := (H^0(U, \mathcal{V}) \otimes_{B_e} B_{\text{st}})^{\Gamma_K}$ over K_0 with the following properties:
   (i) There exists a canonical injective B_{st}-linear Γ_K-equivariant map
$$\alpha_{\mathcal{V}} : D_{\text{st}}(\mathcal{V}) \otimes_E B_{\text{st}} \hookrightarrow H^0(U, \mathcal{V}) \otimes_{B_e} B_{\text{st}}.$$
   (ii) $D_{\text{st}}(\mathcal{V})$ satisfies the inequality
$$\dim_{K_0} D_{\text{st}}(\mathcal{V}) \leq \text{rk}(\mathcal{V})$$
   with equality precisely when $\alpha_{\mathcal{V}}$ is an isomorphism.

(3) Every (φ, N)-module D over K_0 admits a natural isomorphism
$$D \cong ((D \otimes_{K_0} B_{\text{st}})^{\varphi=1,N=0} \otimes_{B_e} B_{\text{st}})^{\Gamma_K}.$$

(4) Given a weakly admissible filtered (φ, N)-module D over K, the vector bundle $\mathcal{F}_{\text{st}}(D)$ on X is trivial with $\text{rk}(\mathcal{F}_{\text{st}}(D)) = \text{rk}(D)$.

THEOREM 1.2.25 (Berger [**Ber02**]). A p-adic Γ_K-representation is de Rham if and only if it is semistable as a Γ_L-representation for some finite extension L of K.

Remark. The main inspiration for Theorem 1.2.25 comes from the *semistable reduction theorem* of Grothendieck [**Gro72**], which states that every abelian variety over K has semistable reduction over a finite extension of K. The proof of this theorem crucially relies on the study of Γ_K-representations with unipotent I_K-action, called *unipotent* Γ_K-representations. For a prime $\ell \neq p$, unipotent ℓ-adic Γ_K-representations serve as analogues of semistable Γ_K-representations; indeed, the ℓ-adic étale cohomology of a proper smooth K-variety with semistable reduction is unipotent. A key fact behind the semistable reduction theorem is that every ℓ-adic Γ_K-representation is unipotent as a Γ_L-representation for some finite extension L of K. Theorem 1.2.25 is an analogue of this fact for p-adic Γ_K-representations, originally conjectured by Fontaine [**Fon94b**].

The key ingredients for Berger's proof of Theorem 1.2.25 are (φ, Γ_∞)-*modules*, which we will briefly discuss in the next section. By means of (φ, Γ_∞)-modules, Berger discovered a remarkable link between de Rham Γ_K-representations and p-adic differential equations. This link allowed Berger to deduce Theorem 1.2.25 from a conjecture of Crew [**Cre98**] on p-adic differential equations proved by André [**And02**], Kedlaya [**Ked04**], and Mebkhout [**Meb02**].

2. Galois representations and φ-modules

In this section, we classify various kinds of Galois representations via certain modules with semilinear endomorphisms. Our primary references for this section are the notes of Brinon-Conrad [**BC**, §3 and §13] and the notes of Fontaine-Ouyang [**FO**, §3].

2.1. Galois representations for fields of characteristic p

Let us begin by introducing some key algebraic notions for this section.

Definition 2.1.1. Let L be an arbitrary field.

(1) A *mod-p Γ_L-representation* is a finite dimensional $\mathbb{F}_p$-vector space V with a continuous homomorphism $\Gamma_L \to \mathrm{GL}(V)$.
(2) An *integrally p-adic Γ_L-representation* is a finitely generated $\mathbb{Z}_p$-module M with a continuous homomorphism $\Gamma_L \to \mathrm{GL}(M)$.

Remark. We may regard mod-p Γ_L-representations as integrally p-adic Γ_L-representations via the natural surjection $\mathbb{Z}_p \twoheadrightarrow \mathbb{F}_p$.

Example 2.1.2. Let L be an arbitrary field.

(1) Given a p-divisible group G over L, its Tate module $T_p(G)$ is an integrally p-adic Γ_L-representation by Proposition 2.1.18 in Chapter II.
(2) For a proper smooth variety X over L, the étale cohomology group $H^n_{\text{ét}}(X_{\overline{E}}, \mathbb{Z}_p)$ is an integrally p-adic Γ_L-representation.

Definition 2.1.3. Given a topological ring R with an action of a group Γ, a *semilinear Γ-module* over R is an R-module M which carries a continuous Γ-action with

$$\gamma(rm) = \gamma(r)\gamma(m) \quad \text{for each } \gamma \in \Gamma,\ r \in R, \text{ and } m \in M.$$

Example 2.1.4. Given a p-adic field K, every p-adic Γ_K-representation V yields a semilinear Γ_K-module $V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$ over B_{dR}.

Definition 2.1.5. Let R be a ring with an endomorphism φ.

(1) Given an R-module M, we define its *φ-twist* to be $\varphi^*(M) := M \otimes_{R,\varphi} R$ where the factor R in the product has φ as structure morphism.
(2) Given two R-modules M and M', we say that an additive map $f : M \to M'$ is *φ-semilinear* if it satisfies the identity

$$f(rm) = \varphi(r)f(m) \quad \text{for each } r \in R \text{ and } m \in M.$$

(3) Given a φ-semilinear map $f : M \to M'$ for R-modules M and M', we define its *linearization* to be the R-linear map $f^{\mathrm{lin}} : \varphi^*(M) \to M'$ with

$$f^{\mathrm{lin}}(m \otimes r) = rf(m) \quad \text{for each } m \in M \text{ and } r \in R.$$

(4) A *φ-module* over R is a finitely generated R-module M with a φ-semilinear endomorphism φ_M, called the *φ-endomorphism* of M and often simply denoted by φ.
(5) An R-linear map $f : M \to M'$ for φ-modules M and M' over R is a *morphism of φ-modules* if it satisfies the identity $f \circ \varphi_M = \varphi_{M'} \circ f$.

Example 2.1.6. Let E be a field of characteristic p with Frobenius endomorphism φ. Every a finite dimensional E-algebra A is naturally a φ-module with φ_A given by the p-th power map. The φ-twist of A and the linearlization of φ_A respectively coincide with the p-Frobenius twist $A^{(p)}$ and the relative p-Frobenius $\varphi_A^{[1]}$.

We aim to classify various Galois representations for a field E of characteristic p. We assume for simplicity that E is perfect and write φ for the Frobenius endomorphism of $\overline{E}$.

Definition 2.1.7. A φ-module D over E is *étale* if φ_D^{lin} is an isomorphism.

Example 2.1.8. A finite flat E-group $G = \operatorname{Spec}(A)$ is étale if and only if the φ-module A is étale by Proposition 1.5.17 in Chapter II and Example 2.1.6.

LEMMA 2.1.9. Let D be an étale φ-module over E.

(1) Given an étale φ-module D' over E, the tensor product $D \otimes_E D'$ is naturally an étale φ-module over E with φ-endomorphism $\varphi_D \otimes \varphi_{D'}$.
(2) The dual $D^\vee$ is naturally an étale φ-module over E with $\varphi_{D^\vee}^{\text{lin}}$ given by the dual map of $(\varphi_D^{\text{lin}})^{-1}$.

PROOF. All statements are straightforward to verify. □

LEMMA 2.1.10. Every semilinear Γ_E-module M over $\overline{E}$ admits a natural isomorphism

$$M \cong M^{\Gamma_E} \otimes_E \overline{E}.$$

PROOF. Let us write $\text{GalMod}_{\overline{E}/E}$ for the category of semilinear Γ_E-modules over $\overline{E}$, where morphisms are Γ_E-equivariant $\overline{E}$-linear maps, and Vect_L for the category of E-vector spaces. A general fact stated in the Stacks Project [**Sta**, Tag 0CDR] yields an equivalence

$$\text{GalMod}_{\overline{E}/E} \cong \text{Vect}_E$$

which sends each $M \in \text{GalMod}_{\overline{E}/E}$ to M^{Γ_E} with the inverse sending each $V \in \text{Vect}_E$ to $V \otimes_E \overline{E}$. Hence we establish the desired assertion. □

PROPOSITION 2.1.11. Let V be a mod-p Γ_E-representation.

(1) The E-vector space $D_{\text{red}}(V) := (V \otimes_{\mathbb{F}_p} \overline{E})^{\Gamma_E}$ is naturally an étale φ-module over E.
(2) There exists a canonical Γ_E-equivariant isomorphism

$$D_{\text{red}}(V) \otimes_E \overline{E} \cong V \otimes_{\mathbb{F}_p} \overline{E} \tag{2.1}$$

which is compatible with the φ-endomorphisms.

PROOF. Lemma 2.1.10 yields the Γ_E-equivariant isomorphism (2.1) and in turn implies that $D_{\text{red}}(V)$ is finite dimensional. We see that $D_{\text{red}}(V)$ is canonically a φ-module over E with φ-endomorphism $1 \otimes \varphi$. Moreover, the isomorphism (2.1) is evidently compatible with the φ-endomorphisms. Hence it remains to prove that $D_{\text{red}}(V)$ is étale.

Let us write $d := \dim_E D_{\text{red}}(V)$. Take an E-basis (e_i) of $D_{\text{red}}(V)$ and an $\mathbb{F}_p$-basis (v_j) of V. For each $i = 1, \cdots, d$, we use the isomorphism (2.1) to obtain an $\overline{E}$-linear relation $e_i = \sum c_{i,j} v_j$ and in turn find $(1 \otimes \varphi)(e_i) = \sum c_{i,j}^p v_j$. Let M and $M^{(p)}$ respectively denote the $d \times d$ matrices whose (i, j)-entries are $c_{i,j}$ and $c_{i,j}^p$. Since M is invertible by construction, $M^{(p)}$ is also invertible by the relation $\det(M^{(p)}) = \det(M)^p$. Now we note that $M^{-1}M^{(p)}$ represents the linearization of $1 \otimes \varphi$ and consequently deduce that $D_{\text{red}}(V)$ is étale as desired. □

Definition 2.1.12. Given a mod-p Γ_E-representation V, we refer to the φ-module $D_{\text{red}}(V)$ in Proposition 2.1.11 as the *mod-p étale φ-module associated to V*.

Remark. If we regard $\overline{E}$ as an $(\mathbb{F}_p, \Gamma_E)$-regular ring, we may identify D_{red} with a refinement of the functor associated to $\overline{E}$. Proposition 2.1.11 implies that every mod-p Γ_E-representation is $\overline{E}$-admissible. Moreover, we can adapt our arguments in Chapter III to prove that D_{red} is compatible with tensor products and duals.

Theorem 2.1.13. The functor D_{red} is an exact equivalence of categories

$$\{ \text{ mod-}p\ \Gamma_E\text{-representations } \} \xrightarrow{\sim} \{ \text{ étale } \varphi\text{-modules over } E \}$$

whose inverse sends each étale φ-module D over E to $V_{\mathrm{red}}(D) := (D \otimes_E \overline{E})^{\varphi=1}$.

Proof. Let D be an étale φ-module over E. The Γ_E-action on $\overline{E}$ is φ-equivariant and induces a Γ_E-action on $V_{\mathrm{red}}(D)$. Moreover, we have a natural $\overline{E}$-linear Γ_E-equivariant map

$$V_{\mathrm{red}}(D) \otimes_{\mathbb{F}_p} \overline{E} \longrightarrow (D \otimes_E \overline{E}) \otimes_{\mathbb{F}_p} \overline{E} \longrightarrow D \otimes_E (\overline{E} \otimes_{\mathbb{F}_p} \overline{E}) \longrightarrow D \otimes_E \overline{E} \tag{2.2}$$

which is evidently compatible with the φ-endomorphisms.

We assert that the map (2.2) is injective. Suppose for contradiction that the kernel is nonzero. Take an $\mathbb{F}_p$-basis (v_i) of $V_{\mathrm{red}}(D)$ and choose a nontrivial $\overline{E}$-linear relation $\sum c_i v_i = 0$ with minimal number of nonzero terms. We may set $c_j = 1$ for some j. We have the identities

$$\sum (\varphi(c_i) - c_i) v_i = \varphi\left(\sum c_i v_i\right) - \sum c_i v_i = 0 \quad \text{ and } \quad \varphi(c_j) - c_j = \varphi(1) - 1 = 0.$$

By the minimality of our relation, each c_i satisfies the equality $c_i = \varphi(c_i)$ and thus lies in $\mathbb{F}_p$. Now we have a nontrivial $\mathbb{F}_p$-linear relation $\sum c_i v_i = 0$ for the $\mathbb{F}_p$-basis (v_i) of $V_{\mathrm{red}}(D)$, thereby obtaining a desired contradiction.

The injectivity of the map (2.2) implies that $V_{\mathrm{red}}(D)$ is finite dimensional over $\mathbb{F}_p$. Moreover, the Γ_E-action on $V_{\mathrm{red}}(D)$ is continuous as each element in $V_{\mathrm{red}}(D)$ is a finite sum of pure tensors and thus has an open stabilizer. We deduce that $V_{\mathrm{red}}(D)$ is a mod-p Γ_E-representation.

Let us now prove that the injective map (2.2) is an isomorphism. We only need to show that $V_{\mathrm{red}}(D)$ has $\mathbb{F}_p$-dimension $d := \dim_E D$, or equivalently that $V_{\mathrm{red}}(D)$ has p^d elements. Lemma 2.1.9 implies that the dual $D^\vee$ of D is naturally an étale φ-module. We take an E-basis (f_i) of $D^\vee$ and see that each f_i yields an E-linear relation $\varphi(f_i) = \sum c_{i,j} f_j$. Let us denote by M the $d \times d$ matrix whose (i, j)-entry is $c_{i,j}$ and set $A := E[t_1, \cdots, t_d]/I$ for the ideal I generated by the polynomials $t_i^p - \sum c_{i,j} t_j$. We obtain an identification

$$V_{\mathrm{red}}(D) \cong \left\{ f \in \mathrm{Hom}_E(D^\vee, \overline{E}) : \varphi \circ f = f \circ \varphi \right\}$$

induced by the isomorphism $D \otimes_E \overline{E} \cong \mathrm{Hom}_E(D^\vee, \overline{E})$ and in turn find

$$V_{\mathrm{red}}(D) \cong \mathrm{Hom}_{E\text{-alg}}(A, \overline{E}).$$

Meanwhile, we observe that A is étale over E; indeed, since M is invertible for representing the linearization of the Frobenius endomorphism on $D^\vee$, the A-module $\Omega_{A/E}$ vanishes by a standard fact stated in the Stacks project [**Sta**, Tag 00RU] and the relation

$$d\Big(t_i^p - \sum c_{i,j} t_j\Big) = -\sum c_{i,j} dt_j \quad \text{for each } i = 1, \cdots, d.$$

Now we note that A has rank p^d over E and consequently see that $V_{\mathrm{red}}(D)$ has p^d elements by a general fact about étale morphisms stated in the Stacks project [**Sta**, Tag 00U3].

The isomorphism (2.2) gives rise to an isomorphism of φ-modules $D \cong D_{\mathrm{red}}(V_{\mathrm{red}}(D))$. Meanwhile, Proposition 2.1.11 shows that every mod-p Γ_E-representation V admits an identification $V \cong V_{\mathrm{red}}(D_{\mathrm{red}}(V))$ and also implies that D_{red} is exact as $\overline{E}$ is faithfully flat over E by a general fact stated in the Stacks project [**Sta**, Tag 00HQ]. Hence we establish the desired assertion. □

Remark. If E is not perfect, both Proposition 2.1.11 and Theorem 2.1.13 remain valid with the separable closure E^{sep} of E in place of $\overline{E}$.

For the rest of this section, we write $\mathscr{E} := W(E)[1/p]$ and $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} := W(\overline{E})$. In addition, we let φ denote the Frobenius automorphism on $\widehat{\mathscr{E}^{\mathrm{un}}} = W(\overline{E})[1/p]$.

LEMMA 2.1.14. The rings $\mathcal{O}_{\mathscr{E}}$ and $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ satisfy the following properties:

(i) There exists a natural Γ_E-action on $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ with $(\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}})^{\Gamma_E} \cong \mathcal{O}_{\mathscr{E}}$.
(ii) Both $\mathcal{O}_{\mathscr{E}}$ and $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ are complete discrete valuation rings with uniformizer p.
(iii) The ring $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ is faithfully flat over $\mathcal{O}_{\mathscr{E}}$.

PROOF. Property (i) is an immediate consequence of Theorem 2.3.1 in Chapter II. Property (ii) is evident by Lemma 2.3.9 in Chapter II. Property (iii) follows from standard facts stated in the Stacks project [**Sta**, Tag 0539 and Tag 00HQ]. □

PROPOSITION 2.1.15. Every finitely generated semilinear Γ_E-module M over $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ admits a natural isomorphism
$$M \cong M^{\Gamma_E} \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}.$$

PROOF. We begin with the case where M is p^n-torsion for some $n \geq 0$. Since the assertion is trivial for $n = 0$, we henceforth assume the inequality $n > 0$ and proceed by induction on n. By Lemma 2.1.10, we observe that $p^{n-1}M$ is isomorphic to $\overline{E}^{\oplus d}$ for some $d \geq 1$ and in turn find $H^1(\Gamma_E, p^{n-1}M) = 0$. Hence we use Lemma 2.1.14 to get a natural commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (p^{n-1}M)^{\Gamma_E} \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} & \longrightarrow & M^{\Gamma_E} \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} & \longrightarrow & (M/p^{n-1}M)^{\Gamma_E} \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & p^{n-1}M & \longrightarrow & M & \longrightarrow & M/p^{n-1}M & \longrightarrow & 0 \end{array}$$

with exact rows. Since the left and right vertical arrows are isomorphisms by Lemma 2.1.10 and the induction hypothesis, the middle vertical arrow is also an isomorphism as desired.

We now consider the general case. For each $i, j \in \mathbb{Z}$ with $i \geq j \geq 1$, our discussion in the previous paragraph and Lemma 2.1.14 together yield a short exact sequence
$$0 \longrightarrow (p^iM/p^jM)^{\Gamma_E} \longrightarrow (M/p^jM)^{\Gamma_E} \longrightarrow (M/p^iM)^{\Gamma_E} \longrightarrow 0.$$
Moreover, Lemma 2.1.14 implies that M admits an identification $M \cong \varprojlim M/p^jM$ and induces a canonical isomorphism $M^{\Gamma_E} \cong \varprojlim (M/p^jM)^{\Gamma_E}$ with surjective transition maps. Hence for each integer $i \geq 1$, we obtain a short exact sequence
$$0 \longrightarrow p^iM^{\Gamma_E} \longrightarrow M^{\Gamma_E} \longrightarrow (M/p^iM)^{\Gamma_E} \longrightarrow 0$$
by a general fact stated in the Stacks project [**Sta**, Tag 03CA] and in turn find
$$M/p^iM \cong (M/p^iM)^{\Gamma_E} \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} \cong M^{\Gamma_E}/p^iM^{\Gamma_E} \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$$
where the first isomorphism follows from our discussion in the previous paragraph. In addition, Lemma 2.1.10 shows that $M^{\Gamma_E}/pM^{\Gamma_E} \cong (M/pM)^{\Gamma_E}$ is finite dimensional over $E \cong \mathcal{O}_{\mathscr{E}}/p\mathcal{O}_{\mathscr{E}}$, which in particular implies that M^{Γ_E} is finitely generated over $\mathcal{O}_{\mathscr{E}}$ by a standard fact stated in the Stacks project [**Sta**, Tag 031D]. Now we establish the desired assertion as the $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$-modules M and $M^{\Gamma_E} \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ are p-adically complete by Lemma 2.1.14. □

Remark. Given a p-adic field K with residue field k, we can show that Proposition 2.1.15 remains valid with K, $\widehat{K^{\mathrm{un}}}$ and k respectively in place of $\mathcal{O}_{\mathscr{E}}$, $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ and E, as remarked after Lemma 2.4.16 in Chapter III.

Definition 2.1.16. A φ-module D over $\mathcal{O}_{\mathscr{E}}$ is *étale* if φ_D^{lin} is an isomorphism.

Remark. We may regard étale φ-modules over E as p-torsion φ-modules over $\mathcal{O}_{\mathscr{E}}$ via the natural surjection $\mathcal{O}_{\mathscr{E}} \twoheadrightarrow E$.

LEMMA 2.1.17. Let D be an étale φ-module over $\mathcal{O}_{\mathscr{E}}$.

(1) Given an étale φ-module D' over $\mathcal{O}_{\mathscr{E}}$, the tensor product $D \otimes_{\mathcal{O}_{\mathscr{E}}} D'$ is naturally an étale φ-module over $\mathcal{O}_{\mathscr{E}}$ with φ-endomorphism $\varphi_D \otimes \varphi_{D'}$.
(2) If D is free, the dual $D^\vee$ is naturally an étale φ-module over $\mathcal{O}_{\mathscr{E}}$ with $\varphi_{D^\vee}^{\mathrm{lin}}$ given by the dual map of $(\varphi_D^{\mathrm{lin}})^{-1}$.

PROOF. All statements are straightforward to verify. □

Remark. If D is torsion, we can show that the Pontryagin dual $D^\wedge := \mathrm{Hom}_{\mathcal{O}_{\mathscr{E}}}(D, \mathscr{E}/\mathcal{O}_{\mathscr{E}})$ is naturally an étale φ-module over $\mathcal{O}_{\mathscr{E}}$. Moreover, for an étale φ-module over E we can naturally identify its E-dual with its Pontryagin dual.

PROPOSITION 2.1.18. Let D be a φ-module over $\mathcal{O}_{\mathscr{E}}$.

(1) For each integer $n \geq 1$, the $\mathcal{O}_{\mathscr{E}}$-modules $p^n D$ and $D/p^n D$ are naturally φ-modules over $\mathcal{O}_{\mathscr{E}}$.
(2) D is étale if and only if D/pD is étale.

PROOF. Statement (1) is straightforward to verify by the equality $\varphi(p) = p$. Let us now consider statement (2). We know by Lemma 2.1.14 that $\mathcal{O}_{\mathscr{E}}$ is a discrete valuation ring with uniformizer p. Since D and its φ-twist are isomorphic $\mathcal{O}_{\mathscr{E}}$-modules, the map φ_D^{lin} is an isomorphism if and only if it is surjective. Hence we establish the desired assertion by observing that the surjectivity of φ_D^{lin} is equivalent to the surjectivity of $\varphi_{D/pD}^{\mathrm{lin}}$. □

PROPOSITION 2.1.19. Let M be an integrally p-adic Γ_E-representation.

(1) The $\mathcal{O}_{\mathscr{E}}$-module $D_{\mathrm{int}}(M) := (M \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}})^{\Gamma_E}$ is naturally an étale φ-module over $\mathcal{O}_{\mathscr{E}}$.
(2) There exists a canonical Γ_E-equivariant isomorphism

$$D_{\mathrm{int}}(M) \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} \cong M \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} \tag{2.3}$$

which is compatible with the φ-endomorphisms.

PROOF. Proposition 2.1.15 yields the Γ_E-equivariant isomorphism (2.5) and in turn implies that $D_{\mathrm{int}}(M)$ is a finitely generated $\mathcal{O}_{\mathscr{E}}$-module. We see that $D_{\mathrm{int}}(M)$ is canonically a φ-module over $\mathcal{O}_{\mathscr{E}}$. Moreover, we note that the isomorphism (2.5) is compatible with the φ-endomorphisms and consequently deduce from Lemma 2.1.14 that $D_{\mathrm{int}}(M)$ is étale as the linearization of the φ-endomorphism on $M \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ is evidently an isomorphism. □

Definition 2.1.20. Given an integrally p-adic Γ_E-representation M, we refer to the φ-module $D_{\mathrm{int}}(M)$ in Proposition 2.1.19 as the *integral étale φ-module associated to M*.

Remark. If we identify mod-p Γ_E-representations and étale φ-modules over E respectively as p-torsion integrally p-adic Γ_E-representations and p-torsion étale φ-modules over $\mathcal{O}_{\mathscr{E}}$, we may regard D_{int} as an extension of D_{red}. Moreover, we can extend the compatibility of D_{red} with tensor products and duals to prove the compatibility of D_{int} with tensor products, duals of free modules, and Pontryagin duals of torsion modules.

THEOREM 2.1.21 (Fontaine [**Fon90**]). The functor D_{int} is an exact equivalence of categories

$$\{ \text{ integrally } p\text{-adic } \Gamma_E\text{-representations } \} \xrightarrow{\sim} \{ \text{ étale } \varphi\text{-modules over } \mathcal{O}_{\mathscr{E}} \}$$

whose inverse sends each étale φ-module D over $\mathcal{O}_{\mathscr{E}}$ to $M_{\mathrm{int}}(D) := (D \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}})^{\varphi=1}$.

PROOF. Let D be an étale φ-module over $\mathcal{O}_{\mathscr{E}}$. The Γ_E-action on $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ is φ-equivariant and induces a Γ_E-action on $M_{\mathrm{int}}(D)$. Moreover, we have a natural $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$-linear Γ_E-equivariant map

$$M_{\mathrm{int}}(D) \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} \longrightarrow (D \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}) \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} \longrightarrow D \otimes_{\mathcal{O}_{\mathscr{E}}} (\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}) \longrightarrow D \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} \quad (2.4)$$

which is evidently compatible with the φ-endomorphisms.

We assert that the map (2.4) is an isomorphism. Let us first consider the case where D is p^n-torsion for some $n \geq 0$. Since the assertion is trivial for $n = 0$, we henceforth assume the inequality $n > 0$ and proceed by induction on n. By Theorem 2.1.13, we observe that $p^{n-1}D \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ is isomorphic to $\overline{E}^{\oplus d}$ for some $d \geq 1$ and in turn find that $\varphi - 1$ is surjective on $p^{n-1}D \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$. Hence we use the snake lemma, Lemma 2.1.14, and Proposition 2.1.18 to obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{\mathrm{int}}(p^{n-1}D) \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} & \longrightarrow & M_{\mathrm{int}}(D) \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} & \longrightarrow & M_{\mathrm{int}}(D/p^{n-1}D) \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & p^{n-1}D \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} & \longrightarrow & D \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} & \longrightarrow & D/p^{n-1}D \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} & \longrightarrow & 0
\end{array}$$

with exact rows. Since the left and right vertical arrows are isomorphisms by Theorem 2.1.13 and the induction hypothesis, the middle vertical arrow is also an isomorphism as desired.

We now consider the general case. For each $i, j \in \mathbb{Z}$ with $i \geq j \geq 1$, our discussion in the previous paragraph and Lemma 2.1.14 together yield a short exact sequence

$$0 \longrightarrow M_{\mathrm{int}}(p^iD/p^jD) \longrightarrow M_{\mathrm{int}}(D/p^jD) \longrightarrow M_{\mathrm{int}}(D/p^iD) \longrightarrow 0.$$

Moreover, Lemma 2.1.14 implies that D admits an identification $D \cong \varprojlim D/p^jD$ and induces a canonical isomorphism $M_{\mathrm{int}}(D) \cong \varprojlim M_{\mathrm{int}}(D/p^jD)$ with surjective transition maps. Hence for each integer $i \geq 1$, we obtain a short exact sequence

$$0 \longrightarrow p^iM_{\mathrm{int}}(D) \longrightarrow M_{\mathrm{int}}(D) \longrightarrow M_{\mathrm{int}}(D/p^iD) \longrightarrow 0$$

by a general fact stated in the Stacks project [**Sta**, Tag 03CA] and in turn find

$$D/p^iD \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} \cong M_{\mathrm{int}}(D/p^iD) \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}} \cong M_{\mathrm{int}}(D)/p^iM_{\mathrm{int}}(D) \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$$

where the first isomorphism follows from our discussion in the previous paragraph. In addition, Theorem 2.1.13 shows that $M_{\mathrm{int}}(D)/pM_{\mathrm{int}}(D) \cong M_{\mathrm{int}}(D/pD)$ is finite dimensional over $\mathbb{F}_p$, which in particular implies that $M_{\mathrm{int}}(D)$ is finitely generated over $\mathbb{Z}_p$ by a standard fact stated in the Stacks project [**Sta**, Tag 031D]. We deduce that the map (2.4) is an isomorphism as the $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$-modules $D \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ and $M_{\mathrm{int}}(D) \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ are p-adically complete by Lemma 2.1.14.

It is not hard to see that the Γ_E-action on $M_{\mathrm{int}}(D)$ is continuous, which means that $M_{\mathrm{int}}(D)$ is an integrally p-adic Γ_E-representation. Moreover, the isomorphism (2.4) induces an isomorphism of φ-modules $D \cong D_{\mathrm{int}}(M_{\mathrm{int}}(D))$. Meanwhile, Proposition 2.1.19 shows that every integrally p-adic Γ_E-representation M admits an identification $M \cong M_{\mathrm{int}}(D_{\mathrm{int}}(M))$ and also implies by Lemma 2.1.14 that D_{int} is exact. Hence we establish the desired assertion. ☐

Remark. If E is not perfect, both Proposition 2.1.19 and Theorem 2.1.21 remain valid with $\mathcal{O}_{\mathscr{E}}$ and $\widehat{\mathcal{O}_{\mathscr{E}}^{\mathrm{un}}}$ replaced by the Cohen rings of E and its separable closure.

Definition 2.1.22. A φ-module D over $\mathscr{E}$ is *étale* if it admits an $\mathcal{O}_{\mathscr{E}}$-lattice which is stable under φ_D and is étale as a φ-module over $\mathcal{O}_{\mathscr{E}}$.

Remark. If D is étale, it is not difficult to show that φ_D^{lin} is an isomorphism.

LEMMA 2.1.23. Given a field L, every p-adic Γ_L-representation V admits a $\mathbb{Z}_p$-lattice M which is stable under the Γ_L-action.

PROOF. Let $\rho : \Gamma_L \to \mathrm{GL}(V)$ denote the map encoding the continuous Γ_L-action on V and choose a $\mathbb{Z}_p$-lattice M_0 in V. Since $\mathrm{GL}(M_0)$ is open in $\mathrm{GL}(V)$, its inverse image Γ_0 under ρ is open in the compact group Γ_L and thus has finite index. Now we take Γ_0-coset representatives $\gamma_1, \cdots, \gamma_n \in \Gamma_K$ and deduce that the $\mathbb{Z}_p$-lattice $M := \sum \rho(\gamma_i) M_0$ in V is stable under the Γ_L-action, thereby completing the proof. □

LEMMA 2.1.24. Let D be an étale φ-module over $\mathscr{E}$.

(1) Given an étale φ-module D' over $\mathscr{E}$, the tensor product $D \otimes_{\mathscr{E}} D'$ is naturally an étale φ-module over $\mathscr{E}$ with φ-endomorphism $\varphi_D \otimes \varphi_{D'}$.
(2) The dual $D^\vee$ is naturally an étale φ-module over $\mathscr{E}$ with $\varphi_{D^\vee}^{\text{lin}}$ given by the dual map of $(\varphi_D^{\text{lin}})^{-1}$.

PROOF. All statements are immediate consequences of Lemma 2.1.17. □

PROPOSITION 2.1.25. Let V be a p-adic Γ_E-representation.

(1) The $\mathscr{E}$-vector space $D_{\text{rat}}(V) := (V \otimes_{\mathbb{Q}_p} \widehat{\mathscr{E}^{\text{un}}})^{\Gamma_E}$ is naturally an étale φ-module over $\mathscr{E}$.
(2) There exists a canonical Γ_E-equivariant isomorphism
$$D_{\text{rat}}(V) \otimes_{\mathscr{E}} \widehat{\mathscr{E}^{\text{un}}} \cong V \otimes_{\mathbb{Q}_p} \widehat{\mathscr{E}^{\text{un}}} \tag{2.5}$$
which is compatible with the φ-endomorphisms.

PROOF. By Lemma 2.1.23, the p-adic Γ_E-representation V admits a $\mathbb{Z}_p$-lattice M which is stable under the Γ_E-action. Moreover, M induces a natural $\mathscr{E}$-linear isomorphism
$$D_{\text{rat}}(V) \cong D_{\text{int}}(M)[1/p].$$
Hence the desired assertions are straightforward to verify. □

Definition 2.1.26. Given a p-adic Γ_E-representation V, we refer to the φ-module $D_{\text{rat}}(V)$ in Proposition 2.1.25 as the *rational étale φ-module associated to V*.

Remark. The compatibility of D_{int} with tensor products and duals of free modules yields the compatibility of D_{rat} with tensor products and duals.

THEOREM 2.1.27. The functor D_{rat} is an exact equivalence of categories
$$\{\, p\text{-adic } \Gamma_E\text{-representations} \,\} \xrightarrow{\sim} \{\, \text{étale } \varphi\text{-modules over } \mathscr{E} \,\}$$
whose inverse sends each étale φ-module D over $\mathscr{E}$ to $V_{\text{rat}}(D) := (D \otimes_{\mathcal{O}_{\mathscr{E}}} \widehat{\mathscr{E}^{\text{un}}})^{\varphi=1}$.

PROOF. For every étale φ-module D over $\mathscr{E}$, we find by Theorem 2.1.21 that $V_{\text{rat}}(D)$ is a p-adic Γ_E-representation and gives rise to an isomorphism of φ-modules $D \cong D_{\text{rat}}(V_{\text{rat}}(D))$. Meanwhile, Proposition 2.1.25 shows that every p-adic Γ_E-representation V admits an identification $V \cong V_{\text{rat}}(D_{\text{rat}}(V))$ and also implies that D_{rat} is exact as $\widehat{\mathscr{E}^{\text{un}}}$ is faithfully flat over $\mathscr{E}$ by a general fact stated in the Stacks project [**Sta**, Tag 00HQ]. Hence we establish the desired assertion. □

Remark. If E is not perfect, both Proposition 2.1.25 and Theorem 2.1.27 remain valid with $\mathscr{E}$ and $\widehat{\mathscr{E}^{\text{un}}}$ replaced by the fraction fields of the Cohen rings of E and its separable closure.

2.2. Galois representations for p-adic fields

In this subsection, we present a classification of various Galois representations for a finite extension K of $\mathbb{Q}_p$. Let us state the following fundamental result without a proof.

THEOREM 2.2.1 (Scholze [**Sch12**]). Let C be a perfectoid field.

(1) Every finite extension of C is perfectoid.
(2) There exists a canonical isomorphism $\Gamma_C \cong \Gamma_{C^\flat}$ induced by a bijection
$$\{ \text{Finite extensions of } C \} \xrightarrow{\sim} \left\{ \text{Finite extensions of } C^\flat \right\}$$
which sends each finite extension L of C to its tilt $L^\flat$.

Definition 2.2.2. Let ζ_{p^∞} denote the set of p-power roots of unity in $\overline{K}$.

(1) The *p-cyclotomic extension* of K is $K_\infty := K(\zeta_{p^\infty})$.
(2) The *completed p-cyclotomic extension* of K is the p-adic completion $\widehat{K_\infty}$ of K_∞.

PROPOSITION 2.2.3. The valued field $\widehat{K_\infty}$ is perfectoid.

PROOF. By Theorem 2.2.1, we may assume the identity $K = \mathbb{Q}_p$. Since $\widehat{K_\infty}$ is evidently complete with a nondiscrete value group, we only need to show that the p-th power map is surjective on $\mathcal{O}_{\widehat{K_\infty}}/p\mathcal{O}_{\widehat{K_\infty}}$. For each $n \geq 1$, we take a primitive p^n-th root of unity $\zeta_{p^n} \in \overline{K}$ and identify the valuation ring of $K(\zeta_{p^n})$ with $\mathbb{Z}_p[\zeta_{p^n}]$. We see that $\mathcal{O}_{\widehat{K_\infty}}$ is isomorphic to the p-adic completion $\widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}$ of $\mathbb{Z}_p[\zeta_{p^\infty}]$. Meanwhile, $\mathbb{Z}_p[\zeta_{p^\infty}]$ admits an identification
$$\mathbb{Z}_p[\zeta_{p^\infty}] \cong \mathbb{Z}_p[t^{1/p^\infty}]/(1 + t + \cdots + t^{p-1})$$
where t^{1/p^∞} denotes the set of p-power roots of the variable t. Hence we find
$$\begin{aligned} \mathcal{O}_{\widehat{K_\infty}}/p\mathcal{O}_{\widehat{K_\infty}} &\cong \widehat{\mathbb{Z}_p[\zeta_{p^\infty}]}/p\widehat{\mathbb{Z}_p[\zeta_{p^\infty}]} \cong \mathbb{Z}_p[\zeta_{p^\infty}]/p\mathbb{Z}_p[\zeta_{p^\infty}] \\ &\cong \mathbb{F}_p[t^{1/p^\infty}]/(t-1)^{p-1} \cong \mathbb{F}_p[t^{1/p^\infty}]/t^{p-1} \end{aligned}$$
and in turn establish the desired assertion. □

Definition 2.2.4. The *perfectoid norm field* of K is $E_K := \widehat{K_\infty}^\flat$.

PROPOSITION 2.2.5. The field $\overline{E_K}$ admits a canonical continuous φ-equivariant Γ_K-action under which Γ_{K_∞} acts via a natural isomorphism $\Gamma_{K_\infty} \cong \Gamma_{E_K}$.

PROOF. The action of $\text{Gal}(K_\infty/K) \cong \Gamma_K/\Gamma_{K_\infty}$ on K_∞ is continuous and uniquely extends to a continuous action on $\widehat{K_\infty}$. Therefore $E_K = \varprojlim_{x \mapsto x^p} \widehat{K_\infty}$ is stable under the continuous Γ_K-action on $\mathbb{C}_K^\flat = \varprojlim_{x \mapsto x^p} \mathbb{C}_K$. Moreover, we naturally identify $\overline{E_K}$ as a subfield of $\mathbb{C}_K^\flat$ by Proposition 3.1.1 in Chapter IV and see that $\overline{E_K}$ is also stable under the Γ_K-action on $\mathbb{C}_K^\flat$. It is evident by construction that the Γ_K-action on $\overline{E_K}$ is φ-eqivariant. Now we obtain a natural isomorphism $\Gamma_{K_\infty} \cong \Gamma_{E_K}$ by Lemma 3.1.11 in Chapter III and Theorem 2.2.1, thereby completing the proof. □

Remark. The work of Fontaine-Wintenberger [**FW79a, FW79b**] constructs a discretely valued subfield E_K^{disc} of E_K, called the *norm field* of K, and proves that Proposition 2.2.5 remains valid with E_K^{disc} and its separable closure respectively in place of E_K and $\overline{E_K}$. Moreover, E_K^{disc} turns out to be isomorphic to $k_\infty((t))$ where k_∞ denotes the residue field of K_∞.

Definition 2.2.6. Let us write $\Gamma_\infty := \mathrm{Gal}(K_\infty/K)$ and $\mathscr{E}_K := W(E_K)[1/p]$.

(1) A (φ, Γ_∞)-ring is a topological $\mathcal{O}_{\mathscr{E}_K}$-algebra R with a φ-semilinear endomorphism φ_R, often simply denoted by φ, and a continuous φ_R-equivariant Γ_∞-action.

(2) Given a (φ, Γ_∞)-ring R, a (φ, Γ_∞)-*module* over R is a φ-module D over R which is also a semilinear Γ_∞-module with the Γ_∞-action being φ-equivariant.

(3) Given a (φ, Γ_∞)-ring R, an R-linear map $f : D \to D'$ for (φ, Γ_∞)-modules D and D' over R is a *morphism of* (φ, Γ_∞)*-modules* if it is Γ_∞-equivariant with $f \circ \varphi_D = \varphi_{D'} \circ f$.

Remark. We can extend the notion of (φ, Γ_∞)-rings by replacing $\mathcal{O}_{\mathscr{E}_K}$ with the Cohen ring of E_K^{disc}. It is worthwhile to mention that many authors denote $\mathrm{Gal}(K_\infty/K)$ by Γ_K and $\mathrm{Gal}(\overline{K}/K)$ by G_K.

Example 2.2.7. Proposition 2.2.5 implies that E_K, $\mathcal{O}_{\mathscr{E}_K}$, and $\mathscr{E}_K$ are naturally (φ, Γ_∞)-rings as Γ_∞ admits an identification $\Gamma_\infty \cong \Gamma_K/\Gamma_{K_\infty}$.

Definition 2.2.8. A (φ, Γ_∞)-module over $\mathcal{O}_{\mathscr{E}_K}$ is *étale* if it is étale as a φ-module.

Lemma 2.2.9. Let D be an étale (φ, Γ_∞)-module over $\mathcal{O}_{\mathscr{E}_K}$.

(1) Given an étale (φ, Γ_∞)-module D' over $\mathcal{O}_{\mathscr{E}_K}$, the tensor product $D \otimes_{\mathcal{O}_{\mathscr{E}_K}} D'$ is naturally an étale (φ, Γ_∞)-module over $\mathcal{O}_{\mathscr{E}_K}$.

(2) If D is free, the dual $D^\vee$ is naturally an étale (φ, Γ_∞)-module over $\mathcal{O}_{\mathscr{E}_K}$.

Proof. The assertions immediately follow from Lemma 2.1.17. □

Remark. If D is torsion, we can show that the Pontryagin dual $D^\wedge := \mathrm{Hom}_{\mathcal{O}_{\mathscr{E}_K}}(D, \mathscr{E}_K/\mathcal{O}_{\mathscr{E}_K})$ is naturally an étale (φ, Γ_∞)-module over $\mathcal{O}_{\mathscr{E}_K}$.

Theorem 2.2.10. There exists an exact equivalence of categories

$$\{ \text{ integrally } p\text{-adic } \Gamma_K\text{-representations } \} \xrightarrow{\sim} \{ \text{ étale } (\varphi, \Gamma_\infty)\text{-modules over } \mathcal{O}_{\mathscr{E}_K} \}$$

induced by the functor D_{int} and its inverse M_{int}.

Proof. Let us write $\widehat{\mathcal{O}_{\mathscr{E}_K}^{\mathrm{un}}} := W(\overline{E_K})$. Consider an integrally p-adic Γ_K-representation M and an étale (φ, Γ_∞)-module D over $\mathcal{O}_{\mathscr{E}_K}$. We observe that M is canonically an integrally p-adic Γ_{E_K}-representation via the isomorphism $\Gamma_{K_\infty} \cong \Gamma_{E_K}$ given by Proposition 2.2.5 and thus gives rise to an étale φ-module $D_{\mathrm{int}}(M) = (M \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}_{\mathscr{E}_K}^{\mathrm{un}}})^{\Gamma_{E_K}}$ over $\mathcal{O}_{\mathscr{E}_K}$. In fact, $D_{\mathrm{int}}(M)$ is naturally an étale (φ, Γ_∞)-module over $\mathcal{O}_{\mathscr{E}_K}$ as easily seen by Proposition 2.2.5 and the isomorphism $\Gamma_\infty \cong \Gamma_K/\Gamma_{K_\infty}$. Meanwhile, since D carries a canonical Γ_K-action given by the surjection $\Gamma_K \twoheadrightarrow \Gamma_\infty$, the $\mathbb{Z}_p$-module $M_{\mathrm{int}}(D) = (D \otimes_{\mathcal{O}_{\mathscr{E}_K}} \widehat{\mathcal{O}_{\mathscr{E}_K}^{\mathrm{un}}})^{\varphi=1}$ is naturally an integrally p-adic Γ_K-representation. Now we apply Theorem 2.1.21 to obtain a $\mathbb{Z}_p$-linear Γ_K-equivariant isomorphism $M \cong M_{\mathrm{int}}(D_{\mathrm{int}}(M))$ and an isomorphism of (φ, Γ_∞)-modules $D \cong D_{\mathrm{int}}(M_{\mathrm{int}}(D))$, thereby establishing the desired assertion. □

Remark. Let us mention two additional facts about the equivalence in Theorem 2.2.10.

(1) By the remark after Definition 2.1.20, the equivalence is compatible with tensor products, duals of free modules, and Pontryagin duals of torsion modules.

(2) By the remark after Theorem 2.1.21, the equivalence remains valid with the Cohen ring of E_K^{disc} in place of $\mathcal{O}_{\mathscr{E}_K}$.

Example 2.2.11. Since the p-adic cyclotomic character χ of K factors through the surjective map $\Gamma_K \twoheadrightarrow \Gamma_\infty$, we use Theorem 2.2.10 to obtain a natural isomorphism of (φ, Γ_∞)-modules

$$D_{\mathrm{int}}(\mathbb{Z}_p(n)) \cong \mathcal{O}_{\mathscr{E}_K}(n) \quad \text{ for each } n \in \mathbb{Z}.$$

Definition 2.2.12. A (φ, Γ_∞)-module over E_K is *étale* if it is étale as a φ-module.

LEMMA 2.2.13. Let D be an étale (φ, Γ_∞)-module over E_K.

(1) Given an étale (φ, Γ_∞)-module D' over E_K, the tensor product $D \otimes_{E_K} D'$ is naturally an étale (φ, Γ_∞)-module over E_K.
(2) The dual $D^\vee$ is naturally an étale (φ, Γ_∞)-module over E_K.

PROOF. The assertions immediately follow from Lemma 2.1.9. □

THEOREM 2.2.14. There exists an exact equivalence of categories

$$\{ \text{ mod-}p\ \Gamma_K\text{-representations } \} \xrightarrow{\sim} \{ \text{ étale } (\varphi, \Gamma_\infty)\text{-modules over } E_K \}$$

induced by the functor D_{red} and its inverse V_{red}.

PROOF. Every mod-p Γ_K-representation is canonically a p-torsion integrally p-adic Γ_K-representation. Meanwhile, every étale (φ, Γ_∞)-module over E_K is naturally a p-torsion étale (φ, Γ_∞)-module over $\mathcal{O}_{\mathscr{E}_K}$. Therefore the desired assertion is straightforward to verify by Theorem 2.2.10. □

Remark. The remark after Theorem 2.2.10 implies the following facts:

(1) The equivalence in Theorem 2.2.14 is compatible with tensor products and duals.
(2) The equivalence in Theorem 2.2.14 remains valid with E_K^{disc} in place of E_K.

Example 2.2.15. Example 2.2.11 and Theorem 2.2.14 together yield a natural isomorphism of (φ, Γ_∞)-modules

$$D_{\text{red}}(\mathbb{F}_p(n)) \cong E_K(n) \quad \text{ for each } n \in \mathbb{Z}.$$

Definition 2.2.16. A (φ, Γ_∞)-module D over $\mathscr{E}_K$ is *étale* if it admits an $\mathcal{O}_{\mathscr{E}_K}$-lattice which is stable under the $\Gamma_\infty[\varphi]$-action and is étale as a φ-module.

LEMMA 2.2.17. Let D be an étale (φ, Γ_∞)-module over $\mathscr{E}_K$.

(1) Given an étale (φ, Γ_∞)-module D' over $\mathscr{E}_K$, the tensor product $D \otimes_{\mathscr{E}_K} D'$ is naturally an étale (φ, Γ_∞)-module over $\mathscr{E}_K$.
(2) The dual $D^\vee$ is naturally an étale (φ, Γ_∞)-module over $\mathscr{E}_K$.

PROOF. The assertions immediately follow from Lemma 2.2.9. □

THEOREM 2.2.18. There exists an exact equivalence of categories

$$\{ \text{ } p\text{-adic } \Gamma_K\text{-representations } \} \xrightarrow{\sim} \{ \text{ étale } (\varphi, \Gamma_\infty)\text{-modules over } \mathscr{E}_K \}$$

induced by the functor D_{rat} and its inverse V_{rat}.

PROOF. Every p-adic Γ_K-representation admits a $\mathbb{Z}_p$-lattice which is stable under the Γ_K-action by Lemma 2.1.23. Meanwhile, every étale (φ, Γ_∞)-module over $\mathscr{E}_K$ contains an $\mathcal{O}_{\mathscr{E}_K}$-lattice which is stable under the $\Gamma_\infty[\varphi]$-action and is étale as a φ-module over $\mathcal{O}_{\mathscr{E}_K}$. Therefore the desired assertion is straightforward to verify by Theorem 2.2.10. □

Remark. The remark after Theorem 2.2.10 implies the following facts:

(1) The equivalence in Theorem 2.2.18 is compatible with tensor products and duals.
(2) The equivalence in Theorem 2.2.18 remains valid with $\mathscr{E}_K$ replaced by the fraction field of the Cohen ring of E_K^{disc}.

Example 2.2.19. Example 2.2.11 and Theorem 2.2.18 together yield a natural isomorphism of (φ, Γ_∞)-modules

$$D_{\text{rat}}(\mathbb{Q}_p(n)) \cong \mathscr{E}_K(n) \quad \text{ for each } n \in \mathbb{Z}.$$

3. The Fargues-Fontaine curve and p-adic geometry

In this section, we introduce some fundamental notions in p-adic geometry and use them to describe another incarnation of the Fargues-Fontaine curve. The primary reference for this section is the book of Scholze-Weinstein [**SW20**].

3.1. Huber rings and adic spaces

Our main objective for this subsection is to briefly discuss a modern framework for nonarchimedean geometry developed by Huber [**Hub93, Hub94**].

PROPOSITION 3.1.1. Every totally ordered abelian group T naturally gives rise to a totally ordered monoid $T \cup \{ 0 \}$ with $\tau \cdot 0 = 0 \cdot \tau = 0$ and $0 \leq \tau$ for each $\tau \in T$.

PROOF. The assertion is straightforward to verify. □

Definition 3.1.2. Given a totally ordered abelian group T, we define its *extension by minimum element* to be the monoid $T \cup \{ 0 \}$ given by Proposition 3.1.1.

Example 3.1.3. For the additive group $\mathbb{R}$ with the natural total order, its extension by minimum element is the additive monoid $[-\infty, \infty)$.

Definition 3.1.4. Let R be a topological ring.

(1) A *valuation* on R is a nonconstant map $v : R \to T \cup \{ 0 \}$ for some totally ordered abelian group T with
$$v(rs) = v(r)v(s) \quad \text{ and } \quad v(r+s) \leq \max(v(r), v(s)) \quad \text{ for every } r, s \in R.$$
(2) Two valuations v and w on R are *equivalent* if there exists an isomorphism of totally ordered monoids $\delta : v(R) \simeq w(R)$ with $\delta(v(r)) = w(r)$ for each $r \in R$.
(3) A valuation $v : R \to T \cup \{ 0 \}$ for a totally ordered abelian group T is *continuous* if for every $\tau \in T$ the set $\{ r \in R : v(r) < \tau \}$ is open in R.
(4) The *continuous valuation spectrum* of R, denoted by $\mathrm{Cont}(R)$, is the set of equivalence classes of continuous valuations on R.
(5) For every $f \in R$ and $x \in \mathrm{Cont}(R)$, the *value of f at x* is $|f(x)| := v(f)$ where v is a representative for x.

Example 3.1.5. We record some simple examples of continuous valuations.

(1) Given a topological ring R, every open prime ideal $\mathfrak{p}$ of R gives rise to a *trivial valuation* $v_{\mathfrak{p}}$ on R with
$$v_{\mathfrak{p}}(r) = \begin{cases} 0 & \text{for } r \in \mathfrak{p}, \\ 1 & \text{for } r \notin \mathfrak{p}. \end{cases}$$
(2) Given a nonarchimedean field L, its valuation ν is evidently continuous.

Remark. For a nonarchimedean field L, we may take values of the valuation ν in the multiplicative monoid $[0, \infty)$ via an isomorphism $[0, \infty) \simeq (-\infty, \infty]$ given by a logarithm map.

LEMMA 3.1.6. Given a topological ring R, every valuation v on R satisfies the equalities
$$v(0) = 0 \quad \text{ and } \quad v(1) = 1.$$

PROOF. We note that every nonzero element in $v(R)$ is invertible. If $v(0)$ is nonzero, for each $r \in R$ we have $v(0) = v(r)v(0)$ and thus obtain the identity $v(r) = 1$, which is impossible as v is not constant. Hence we deduce that $v(0)$ is zero. Moreover, we take an element $r \in R$ with $v(r) \neq 0$ and use the relation $v(r) = v(r)v(1)$ to find $v(1) = 1$. □

Definition 3.1.7. Let R be a topological ring.

(1) R is *adic* if it admits a neighborhood basis at 0 given by the powers of an ideal I, called an *ideal of definition*.
(2) R is *Huber* if it admits an open adic subring R_0, called a *ring of definition*, with a finitely generated ideal of definition.

Example 3.1.8. Let us present some simple examples of Huber rings.

(1) Every ring R with the discrete topology is a Huber ring; indeed, R is a ring of definition with an ideal of definition given by the zero ideal.
(2) Every nonarchimedean field L is a Huber ring; indeed, $\mathcal{O}_L$ is a ring of definition with an ideal of definition given by $m\mathcal{O}_L$ for any element m in the maximal ideal.

Definition 3.1.9. Let R be a Huber ring.

(1) A *pair of definition* for R is a pair (R_0, I) for some ring of definition R_0 and its ideal of definition I.
(2) A *rational pair* for R is a pair (T, g) consisting of a nonempty finite set $T \subseteq R$ with TR being open in R and an element $g \in R$ with $g^n \neq 0$ for every integer $n > 0$.

PROPOSITION 3.1.10. Let R be a Huber ring and (T, g) be a rational pair for R. Fix a pair of definition (R_0, I) for R and denote the elements of T by $f_1, \cdots, f_n$.

(1) The ring $R_0[T/g] := R_0[f_1/g, \cdots, f_n/g]$ is adic with an ideal of definition $IR_0[T/g]$.
(2) The ring $R[1/g]$ is naturally a Huber ring with a pair of definition $(R_0[T/g], IR_0[T/g])$.

PROOF. Let us consider the topology on $R[1/g]$ with a neighborhood basis at 0 given by the powers $J := IR_0[T/g]$. It suffices to show that the multiplication on $R[1/g]$ is continuous. Take an arbitrary intger $n > 0$. We only need to find an integer $m > 0$ with $g^{-1}J^m \subseteq J^n$ as the multiplication on R is continuous.

Fix an integer $i > 0$ with $I^i \subseteq TR$ and choose a finite set $S \subseteq R$ such that TS contains generators of I^i over R_0. There exists an integer $j > 0$ with $SI^j \subseteq I^n$ by the continuity of the multiplication on R. Hence we set $m = i + j$ and find

$$g^{-1}I^m = g^{-1}I^{i+j} \subseteq g^{-1}TSI^j \subseteq g^{-1}TI^n \subseteq J^n,$$

thereby establishing the desired assertion. □

Definition 3.1.11. Let R be a Huber ring with a pair of definition (R_0, I) and (T, g) be a rational pair for R.

(1) The *localization* of (R_0, I) with respect to (T, g) is the pair $(R_0[T/g], IR_0[T/g])$ given by Proposition 3.1.10.
(2) The *localization* of R with respect to (T, g), denoted by $R(T/g)$, is the Huber ring $R[1/g]$ with a pair of definition $(R_0[T/g], IR_0[T/g])$.

Example 3.1.12. Let L be a nonarchimedean field and (T, g) be a rational pair for L.

(1) If $\mathcal{O}_L$ contains f/g for each $f \in T$, the localization $L(T/g)$ is the field L with its valuation topology; indeed, for every element m in the maximal ideal, the localization of $(\mathcal{O}_L, m\mathcal{O}_L)$ is $(\mathcal{O}_L, m\mathcal{O}_L)$.
(2) If $\mathcal{O}_L$ does not contain f/g for some $f \in T$, the localization $L(T/g)$ is the field L with the trivial topology; indeed, for every element m in the maximal ideal, the localization of $(\mathcal{O}_L, m\mathcal{O}_L)$ is (L, L).

Definition 3.1.13. Let R be a topological ring.

(1) A subset S of R is *bounded* if for every open neighborhood U of 0 there exists an open neighborhood V of 0 with $VS \subseteq U$.
(2) An element $r \in R$ is *power-bounded* if its powers form a bounded subset of R.

Example 3.1.14. Let us describe the power-bounded elements for some Huber rings.

(1) For a ring R with the discrete topology, every element in R is power-bounded; indeed, every subset of R is bounded as the zero ideal is open.
(2) For a nonarchimedean field L, the power-bounded elements in L are precisely the elements in the valuation ring $\mathcal{O}_L$; indeed, a subset of L is bounded if and only if its image under the valuation is bounded.

Definition 3.1.15. A *Huber pair* is a pair (R, R^+) consisting of a Huber ring R and its integrally closed open subring R^+ such that all elements in R^+ are power-bounded.

Example 3.1.16. Below are Huber pairs given by Example 3.1.14.

(1) Every ring R with the discrete topology yields a Huber pair (R, R).
(2) Every nonarchimedean field L yields a Huber pair $(L, \mathcal{O}_L)$.

Definition 3.1.17. Let (R, R^+) be a Huber pair.

(1) The *adic spectrum* of (R, R^+) is the set
$$\mathrm{Spa}(R, R^+) := \left\{\, x \in \mathrm{Cont}(R) : |f(x)| \leq 1 \text{ for each } f \in R^+ \,\right\}$$
with the topology generated by subsets of the form
$$\mathcal{U}(f/g) := \left\{\, x \in \mathrm{Spa}(R, R^+) : |f(x)| \leq |g(x)| \neq 0 \,\right\} \qquad \text{for some } f, g \in R.$$
(2) Given a rational pair (T, g) for R, the associated *rational subset* of $\mathrm{Spa}(R, R^+)$ is
$$\mathcal{U}(T/g) := \left\{\, x \in \mathrm{Spa}(R, R^+) : |f(x)| \leq |g(x)| \neq 0 \text{ for each } f \in T \,\right\}.$$

Remark. Huber [**Hub94**] shows that the topological space $\mathrm{Spa}(R, R^+)$ is spectral.

Example 3.1.18. Given a nonarchimedean field L, we assert that $\mathrm{Spa}(L, \mathcal{O}_L)$ consists of a unique point given by the valuation ν on L. Let w be a continuous valuation on L whose equivalence class lies in $\mathrm{Spa}(L, \mathcal{O}_L)$. We note that $\mathcal{O}_L$ lies in the ring
$$\mathcal{O}_w := \{\, c \in L : w(c) \leq 1 \,\}.$$
If $\mathcal{O}_w$ contains an element $c \notin \mathcal{O}_L$, we find $\mathcal{O}_w \supseteq \mathcal{O}_L[c] = L$ and deduce that w is trivial, which is impossible since the zero ideal is not open in L. Hence we must have $\mathcal{O}_w = \mathcal{O}_L$. Now we see that w is equivalent to ν via the isomorphism
$$w(L^\times) \cong L^\times/\mathcal{O}_w^\times = L^\times/\mathcal{O}_L^\times \cong \nu(L^\times),$$
thereby completing the proof.

Definition 3.1.19. Given two Huber pairs (R, R^+) and (Q, Q^+), a *morphism* from (R, R^+) to (Q, Q^+) is a continuous ring homomorphism $h : R \to Q$ with $h(R^+) \subseteq Q^+$.

LEMMA 3.1.20. For Huber pairs (R, R^+) and (Q, Q^+), every morphism $h : (R, R^+) \to (Q, Q^+)$ naturally induces a continuous map $\eta : \mathrm{Spa}(Q, Q^+) \to \mathrm{Spa}(R, R^+)$.

PROOF. Take an arbitrary point $x \in \mathrm{Spa}(Q, Q^+)$ and choose its representative v. The equivalence class of the continuous valuation $v \circ h$ lies in $\mathrm{Spa}(R, R^+)$. Moreover, this equivalence class belongs to $\mathcal{U}(f/g)$ for some $f, g \in R$ if and only if x belongs to $\mathcal{U}(h(f)/h(g))$. Therefore we establish the desired assertion. $\square$

In order to describe additional structures on the adic spectrum of a Huber pair, we state the following crucial result without a proof.

PROPOSITION 3.1.21. Let (R, R^+) be a Huber pair and write $\mathcal{S} := \mathrm{Spa}(R, R^+)$.

(1) The rational subsets of $\mathcal{S}$ form a basis of open sets in $\mathcal{S}$.
(2) Every rational subset $\mathcal{U}$ of $\mathcal{S}$ naturally gives rise a Huber pair $(\mathcal{O}_\mathcal{S}(\mathcal{U}), \mathcal{O}_\mathcal{S}^+(\mathcal{U}))$ and a morphism $h_\mathcal{U} : (R, R^+) \to (\mathcal{O}_\mathcal{S}(\mathcal{U}), \mathcal{O}_\mathcal{S}^+(\mathcal{U}))$ with the following properties:
   (i) Given a rational pair (T, g) for R with $\mathcal{U} = \mathcal{U}(T/g)$, the Huber ring $\mathcal{O}_\mathcal{S}(\mathcal{U})$ is canonically isomorphic to the completion of the localization $R(T/g)$.
   (ii) The ring $\mathcal{O}_\mathcal{S}^+(\mathcal{U})$ admits an identity
$$\mathcal{O}_\mathcal{S}^+(\mathcal{U}) = \{ f \in \mathcal{O}_\mathcal{S}(\mathcal{W}) : |f(\mathcal{U})| \leq 1 \text{ for each } x \in \mathcal{W} \}.$$
   (iii) The map $\mathrm{Spa}(\mathcal{O}_\mathcal{S}(\mathcal{U}), \mathcal{O}_\mathcal{S}^+(\mathcal{U})) \to \mathcal{S}$ induced by $h_\mathcal{U}$ is a homeomorphism onto $\mathcal{U}$.
   (iv) A morphism of Huber pairs $(R, R^+) \to (Q, Q^+)$ with Q being complete uniquely factors through $(\mathcal{O}_\mathcal{S}(\mathcal{U}), \mathcal{O}_\mathcal{S}^+(\mathcal{U}))$ if the map $\mathrm{Spa}(Q, Q^+) \to \mathcal{S}$ factors through $\mathcal{U}$.

Remark. Curious readers can find a proof of Proposition 3.1.21 in the original article of Huber [**Hub94**, Proposition 1.3] or the survey article of Wedhorn [**Wed19**, Proposition 8.2].

Definition 3.1.22. Let (R, R^+) be a Huber pair and write $\mathcal{S} := \mathrm{Spa}(R, R^+)$.

(1) Given a rational subset $\mathcal{U}$ of $\mathcal{S}$, its *affinoid Huber pair* is the pair $(\mathcal{O}_\mathcal{S}(\mathcal{U}), \mathcal{O}_\mathcal{S}^+(\mathcal{U}))$ constructed in Proposition 3.1.21.
(2) The *structure presheaf* on $\mathcal{S}$ is the presheaf $\mathcal{O}_\mathcal{S}$ of topological rings with
$$\mathcal{O}_\mathcal{S}(\mathcal{W}) = \varprojlim_{\substack{\mathcal{U} \subseteq \mathcal{W} \\ \mathcal{U} \text{ rational}}} \mathcal{O}_\mathcal{S}(\mathcal{U}) \quad \text{for every open } \mathcal{W} \subseteq \mathcal{S}.$$
(3) The pair (R, R^+) is *sheafy* if $\mathcal{O}_\mathcal{S}$ is a sheaf of topological rings.

Remark. We can also define the presheaf $\mathcal{O}_\mathcal{S}^+$ of topological rings with
$$\mathcal{O}_\mathcal{S}^+(\mathcal{W}) = \varprojlim_{\substack{\mathcal{U} \subseteq \mathcal{W} \\ \mathcal{U} \text{ rational}}} \mathcal{O}_\mathcal{S}^+(\mathcal{U}) \quad \text{for every open } \mathcal{W} \subseteq \mathcal{S}.$$
If (R, R^+) is sheafy, $\mathcal{O}_\mathcal{S}^+$ is a sheaf by Proposition 3.1.21.

Example 3.1.23. We present sheafy Huber pairs given by a result of Huber [**Hub94**].

(1) Every Huber pair (R, R^+) for a ring R with the discrete topology is sheafy.
(2) Every Huber pair (R, R^+) for a complete Huber ring R with a noetherian ring of definition is sheafy.

Definition 3.1.24. A *locally valued ringed space* is a topological space $\mathcal{S}$ with a sheaf $\mathcal{O}_\mathcal{S}$ of topological rings and an element $v_x \in \mathrm{Cont}(\mathcal{O}_{\mathcal{S},x})$ for each $x \in \mathcal{S}$.

PROPOSITION 3.1.25. Given a sheafy Huber pair (R, R^+), its adic spectrum $\mathcal{S}$ is naturally a locally valued ringed space.

PROOF. The presheaf $\mathcal{O}_\mathcal{S}$ is a sheaf as (R, R^+) is sheafy. Moreover, by Proposition 3.1.21, every $x \in \mathcal{S}$ yields an element $v_\mathcal{U} \in \mathrm{Cont}(\mathcal{O}_\mathcal{S}(\mathcal{U}))$ for each rational subset $\mathcal{U}$ of $\mathcal{S}$ with $x \in \mathcal{U}$ and consequently gives rise to an element $v_x \in \mathrm{Cont}(\mathcal{O}_{\mathcal{S},x})$. □

Remark. It turns out that every stalk of $\mathcal{O}_\mathcal{S}$ is a local ring.

Definition 3.1.26. An *adic space* is a locally valued ringed space which admits an open cover by adic spectra of sheafy Huber pairs.

3.2. The adic Fargues-Fontaine curve

Throughout this subsection, we let F be an algebraically closed perfectoid field of characteristic p and denote by Y the set of equivalence classes of untilts of F in characteristic 0. In addition, we write $\mathfrak{m}_F$ for the maximal ideal of $\mathcal{O}_F$ and $|\mathfrak{m}_F|$ for the image of $\mathfrak{m}_F \setminus \{ 0 \}$ in the value group of F.

PROPOSITION 3.2.1. The topological ring A_{inf} yields a Huber pair $(A_{\text{inf}}, A_{\text{inf}})$.

PROOF. The ring A_{inf} is adic and Huber; indeed, for every nonzero $\varpi \in \mathfrak{m}_F$, it admits an ideal of definition generated by p and $[\varpi]$. We see that every subset of A_{inf} is bounded and consequently establish the desired assertion. □

LEMMA 3.2.2. Given a nonzero element $\varpi \in \mathfrak{m}_F$, a valuation v on A_{inf} with $v([\varpi]) = 0$ satisfies the equality $v([m]) = 0$ for every $m \in \mathfrak{m}_F$.

PROOF. The assertion is straightforward to verify by observing that every $m \in \mathfrak{m}_F$ yields an integer $n \geq 1$ with $m^n \in \varpi\mathcal{O}_F$. □

Definition 3.2.3. The *adic punctured disk of untilts* is the set

$$\mathcal{Y} = \mathcal{Y}_F := \{ x \in \text{Spa}(A_{\text{inf}}, A_{\text{inf}}) : |p[\varpi](x)| \neq 0 \}$$

where we fix a nonzero element $\varpi \in \mathfrak{m}_F$.

Remark. Lemma 3.2.2 shows that $\mathcal{Y}$ does not depend on the choice of ϖ.

PROPOSITION 3.2.4. Let C be an untilt of F in characteristic 0.

(1) The nonarchimedean field C yields a Huber pair $(C, \mathcal{O}_C)$.
(2) There exists a natural continuous map $\theta_C^{\text{ad}} : \text{Spa}(C, \mathcal{O}_C) \to \text{Spa}(A_{\text{inf}}, A_{\text{inf}})$ induced by the Fontaine map θ_C.

PROOF. Since statement (1) is evident by Example 3.1.16, we only need to establish statement (2). It is straightforward to verify that the map $\theta_C : A_{\text{inf}} \twoheadrightarrow \mathcal{O}_C$ is continuous. Therefore the composition of θ_C with the embedding $\mathcal{O}_C \hookrightarrow C$ yields a morphism of Huber pairs $(A_{\text{inf}}, A_{\text{inf}}) \to (C, \mathcal{O}_C)$. Now the desired assertion follows from Lemma 3.1.20. □

Definition 3.2.5. For an untilt C of F in characteristic 0, we refer to the map θ_C^{ad} constructed in Proposition 3.2.4 as the *adic Fontaine map* of C.

PROPOSITION 3.2.6. There exists a natural embedding $Y \hookrightarrow \mathcal{Y}$ which sends each $y \in Y$ with a representative C to the image of θ_C^{ad}.

PROOF. For each untilt C of F in characteristic 0, we apply Example 3.1.18 to see that the image of the map $\theta_C^{\text{ad}} : \text{Spa}(C, \mathcal{O}_C) \to \text{Spa}(A_{\text{inf}}, A_{\text{inf}})$ is a point in $\mathcal{Y}$; indeed, it admits a representative v_C with $v_C(f) = |\theta_C(f)|_C$ for every $f \in A_{\text{inf}}$. If two untilts C and C' of F in characteristic 0 are equivalent, it is straightforward to verify that v_C and $v_{C'}$ are equivalent. Conversely, if the valuations v_C and $v_{C'}$ for untilts C and C' of F in characteristic 0 are equivalent, Theorem 1.1.27 in Chapter IV implies that C and C' are equivalent. Therefore we obtain the desired assertion. □

Remark. Similarly, there exists a natural embedding $(0, 1) \hookrightarrow \mathcal{Y}$ which sends each $\rho \in (0, 1)$ to the equivalence class of the Gauss ρ-norm on A_{inf}. Moreover, the image of this map is disjoint from the image of the embedding $Y \hookrightarrow \mathcal{Y}$.

Definition 3.2.7. Given an element $y \in Y$, its associated *classical point* on $\mathcal{Y}$ is the image of y under the embedding $Y \hookrightarrow \mathcal{Y}$ given by Proposition 3.2.6.

LEMMA 3.2.8. Two elements $c_1, c_2 \in \mathcal{O}_F$ with $|c_1| \leq |c_2|$ satisfy the inequality

$$|[c_1](x)| \leq |[c_2](x)| \qquad \text{for every } x \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) \tag{3.1}$$

with equality precisely when we have $|c_1| = |c_2|$.

PROOF. Since $c_1 c_2^{-1}$ lies in $\mathcal{O}_F$, the inequality (3.1) follows from the identity

$$|[c_1](x)| = \left|[c_1 c_2^{-1}](x)\right| |[c_2](x)|\,.$$

If we have $|c_1| = |c_2|$, we find $|[c_1](x)| \geq |[c_2](x)|$ by the relation $c_1^{-1}c_2 \in \mathcal{O}_F$ and thus see that the inequality (3.1) is an equality. If we have $|c_1| < |c_2|$, we note that $[c_1^{-1}c_2] \in A_{\mathrm{inf}}$ is topologically nilpotent and in turn use Lemma 3.1.6 to obtain the relation $\left|[c_1 c_2^{-1}](x)\right| < 1$, which in particular implies that the inequality (3.1) is strict. Therefore we establish the desired assertion. □

Definition 3.2.9. Given elements $a, b \in |\mathfrak{m}_F|$, the *adic* $[a,b]$*-annulus of untilts* is

$$\mathcal{Y}_{[a,b]} := \{\, x \in \mathcal{Y} : |[\varpi_a](x)| \leq |p(x)| \leq |[\varpi_b](x)| \,\}$$

where we fix elements $\varpi_a, \varpi_b \in \mathfrak{m}_F$ with $|\varpi_a| = a$ and $|\varpi_b| = b$.

Remark. Lemma 3.2.8 implies that $\mathcal{Y}_{[a,b]}$ does not depend on the choice of ϖ_a and ϖ_b.

PROPOSITION 3.2.10. Let a, b be elements in $|\mathfrak{m}_F|$.

(1) If a and b satisfy the inequality $a > b$, the set $\mathcal{Y}_{[a,b]}$ is empty.
(2) For each $a', b' \in |\mathfrak{m}_F|$ with $[a,b] \subseteq [a',b']$, we have $\mathcal{Y}_{[a,b]} \subseteq \mathcal{Y}_{[a',b']}$.
(3) For each $a', b' \in |\mathfrak{m}_F|$ with $[a,b]$ and $[a',b']$ being disjoint, $\mathcal{Y}_{[a,b]}$ and $\mathcal{Y}_{[a',b']}$ are disjoint.

PROOF. The assertions are straightforward to verify by Lemma 3.2.8. □

Remark. If a and b are arbitrary elements in the interval $(0,1)$, Proposition 3.2.10 allows us to extend Definition 3.2.9 by setting $\mathcal{Y}_{[a,b]}$ as the intersection of the sets $\mathcal{Y}_{[a',b']}$ with $a', b' \in |\mathfrak{m}_F|$ and $[a,b] \subseteq [a',b']$.

PROPOSITION 3.2.11. Given an element $y \in Y_{[a,b]}$ for some $a, b \in |\mathfrak{m}_F|$, its associated classical point on $\mathcal{Y}$ lies in $\mathcal{Y}_{[a,b]}$.

PROOF. Choose a representative C of y. The classical point associated to y admits a representative v_C with $v_C(f) = |\theta_C(f)|_C$ for every $f \in A_{\mathrm{inf}}$. Hence we deduce the assertion by observing the identities $v_C(p) = |y|$ and $v_C([\varpi]) = |\varpi|$ for every $\varpi \in \mathfrak{m}_F$. □

Remark. Similarly, for each $\rho \in [a,b]$ the equivalence class of the Gauss ρ-norm on A_{inf} yields a point in $\mathcal{Y}_{[a,b]}$.

PROPOSITION 3.2.12. The set $\mathcal{Y}$ admits an identity

$$\mathcal{Y} = \bigcup_{a,b \in |\mathfrak{m}_F|} \mathcal{Y}_{[a,b]}.$$

PROOF. Take an arbitrary point $x \in \mathcal{Y}$ and choose a nonzero element $\varpi \in \mathfrak{m}_F$. Since both p and $[\varpi]$ are topologically nilpotent in A_{inf}, we apply Lemma 3.1.6 to obtain the inequalities $|p(x)| < 1$ and $|[\varpi](x)| < 1$. Hence we find integers $i, j > 0$ with

$$\left|[\varpi^i](x)\right| \leq |p(x)| \leq \left|[\varpi^{1/p^j}](x)\right|$$

and in turn deduce that x lies in $\mathcal{Y}_{[|\varpi|^i, |\varpi|^{1/p^j}]}$. Now the desired assertion is evident. □

Let us now invoke the following technical result without a proof.

THEOREM 3.2.13 (Kedlaya-Liu [**KL15**]). Let a, b be elements in $|\mathfrak{m}_F|$. For every integrally closed open subring $B^+_{[a,b]}$ of $B_{[a,b]}$, the pair $(B_{[a,b]}, B^+_{[a,b]})$ is a sheafy Huber pair.

PROPOSITION 3.2.14. Let a, b be elements in $|\mathfrak{m}_F|$.

(1) $\mathcal{Y}_{[a,b]}$ is a rational subset of $\mathcal{S} := \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$.
(2) $\mathcal{Y}_{[a,b]}$ is naturally an adic space with a canonical isomorphism $\mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[a,b]}) \cong B_{[a,b]}$.

PROOF. Let us begin with statement (1). Take $\varpi_a, \varpi_b \in \mathfrak{m}_F$ with $|\varpi_a| = a$ and $|\varpi_b| = b$. We note that $\mathcal{Y}_{[a,b]}$ admits an identification

$$\mathcal{Y}_{[a,b]} = \left\{ x \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) : |[\varpi_a\varpi_b](x)|, \left|p^2(x)\right| \leq |[\varpi_b]p(x)| \neq 0 \right\}.$$

Since the set $T := \left\{ [\varpi_a\varpi_b], p^2 \right\}$ generates an open ideal in A_{inf}, we deduce that $\mathcal{Y}_{[a,b]}$ coincides with the rational subset $\mathcal{U}(T/[\varpi_b]p)$ of $\mathcal{S} = \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ as desired.

It remains to prove statement (2). In light of Theorem 3.2.13, it suffices to estabish a natural isomorphism $\mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[a,b]}) \cong B_{[a,b]}$. Proposition 3.1.21 shows that $\mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[a,b]})$ is canonically isomorphic to the completion of $A_{\mathrm{inf}}(T/[\varpi_b]p)$. Since A_{inf} is adic with an ideal of definition generated by p and $[\varpi_a]$, we may identify $A_{\mathrm{inf}}(T/[\varpi_b]p)$ as the ring $A_{\mathrm{inf}}[1/p, 1/[\varpi_b]]$ with the p-adic topology. By Lemma 3.1.9 in Chapter IV, it is enough to prove the identity

$$A_{\mathrm{inf}}[[\varpi_a]/p, p/[\varpi_b]] = \{ f \in A_{\mathrm{inf}}[1/p, 1/[\varpi_b]] : |f|_a \leq 1 \text{ and } |f|_b \leq 1 \}.$$

Every $f \in A_{\mathrm{inf}}[[\varpi_a]/p, p/[\varpi_b]]$ evidently satisfies the inequalities $|f|_a \leq 1$ and $|f|_b \leq 1$. Hence we only need to show that every $f \in A_{\mathrm{inf}}[1/p, 1/[\varpi_b]]$ with $|f|_a \leq 1$ and $|f|_b \leq 1$ lies in $A_{\mathrm{inf}}[[\varpi_a]/p, p/[\varpi_b]]$. By Proposition 1.2.3 in Chapter IV, we may write $f = \sum [c_n]p^n$ with $c_n \in F$ and take an integer $m > 0$ with $\varpi_b^m c_n \in \mathcal{O}_F$ for each $n \in \mathbb{Z}$. We find

$$\sum_{n \geq m} [c_n]p^n = (p/[\varpi_b])^m \sum_{n \geq 0} [\varpi_b^m c_{n+m}]p^n \in A_{\mathrm{inf}}[p/[\varpi_b]].$$

Meanwhile, as we have the inequalities

$$|c_n| \, |\varpi_a|^n \leq |f|_a \leq 1 \quad \text{and} \quad |c_n| \, |\varpi_b|^n \leq |f|_b \leq 1 \quad \text{for each } n \in \mathbb{Z},$$

we obtain the relation

$$\sum_{n<m} [c_n]p^n = \sum_{n<0} [c_n\varpi_a^n] \cdot ([\varpi_a]/p)^{-n} + \sum_{0 \leq n < m} [c_n\varpi_b^n] \cdot (p/[\varpi_b])^n \in A_{\mathrm{inf}}[[\varpi_a]/p, p/[\varpi_b]].$$

Now the desired assertion is evident. □

Remark. We state two additional facts about the adic space $\mathcal{Y}_{[a,b]}$ and the Huber ring $B_{[a,b]}$, proved by Kedlaya [**Ked05, Ked16**] and Fargues-Fontaine [**FF18**].

(1) The space $\mathcal{Y}_{[a,b]}$ is noetherian.
(2) The ring $B_{[a,b]}$ is a principal ideal domain and gives rise to a natural bijectoin between the maximal ideals of $B_{[a,b]}$ and the classical points on $\mathcal{Y}_{[a,b]}$.

PROPOSITION 3.2.15. The set $\mathcal{Y}$ is open in $\mathcal{S} := \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ and is naturally an adic space with a canonical isomorphism $\mathcal{O}_{\mathcal{Y}}(\mathcal{Y}) \cong B$.

PROOF. Since every rational subset of $\mathcal{S} = \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ is open, the assertion follows from Proposition 3.2.10, Proposition 3.2.12 and Proposition 3.2.14. □

Remark. For each $y \in Y$, we can use Proposition 3.2.15 to identify $B^+_{\mathrm{dR}}(y)$ with the completed local ring at the associated classical point.

Proposition 3.2.16. There exists a canonical homeomorphism $\phi : \mathcal{Y} \to \mathcal{Y}$ induced by the Frobenius automorphism of A_{inf}.

Proof. The Frobenius automorphism φ_{inf} on A_{inf} is a topological automorphism, as easily seen by the relations $\varphi_{\text{inf}}(p) = p$ and $\varphi_{\text{inf}}([\varpi]) = [\varpi^p])$ for each $\varpi \in \mathfrak{m}_F$. Hence φ_{inf} yields an automorphism of the Huber pair $(A_{\text{inf}}, A_{\text{inf}})$ and in turn gives rise to a homeomorphism $\varphi_{\text{inf}}^{\text{ad}} : \text{Spa}(A_{\text{inf}}, A_{\text{inf}}) \to \text{Spa}(A_{\text{inf}}, A_{\text{inf}})$ by Lemma 3.1.20. Now we observe that $\varphi_{\text{inf}}^{\text{ad}}$ restricts to a homeomorphism $\phi : \mathcal{Y} \to \mathcal{Y}$, thereby completing the proof. □

Definition 3.2.17. We refer to the homeomorphism ϕ constructed in Proposition 3.2.16 as the *Frobenius action* on $\mathcal{Y}$ and define the *adic Fargues-Fontaine curve* to be the set

$$\mathcal{X} = \mathcal{X}_F := \mathcal{Y}/\phi^{\mathbb{Z}}.$$

Remark. It is not hard to see that the Frobenius action on $\mathcal{Y}$ restricts to the Frobenius action on Y via the embedding $Y \hookrightarrow \mathcal{Y}$ given by Proposition 3.2.6.

Proposition 3.2.18. For every $a, b \in |\mathfrak{m}_F|$, the Frobenius action ϕ on $\mathcal{Y}$ naturally induces a homeomorphism $\mathcal{Y}_{[a^p, b^p]} \to \mathcal{Y}_{[a,b]}$ and a sheaf isomorphism $\mathcal{O}_{\mathcal{Y}_{[a^p,b^p]}} \cong \mathcal{O}_{\mathcal{Y}_{[a,b]}}$.

Proof. It is straightforward to verify that ϕ maps $\mathcal{Y}_{[a^p,b^p]}$ homeomorphically onto $\mathcal{Y}_{[a,b]}$. Moreover, since $\mathcal{Y}_{[a,b]}$ and $\mathcal{Y}_{[a^p,b^p]}$ are rational subset of $\mathcal{S} := \text{Spa}(A_{\text{inf}}, A_{\text{inf}})$ with canonical isomorphisms $\mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[a,b]}) \cong B_{[a,b]}$ and $\mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[a^p,b^p]}) \cong B_{[a^p,b^p]}$ as noted in Proposition 3.2.14, we apply Proposition 1.2.18 in Chapter IV to see that ϕ induces a natural isomorphism

$$(\mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[a,b]}), \mathcal{O}_{\mathcal{S}}^+(\mathcal{Y}_{[a,b]})) \cong (\mathcal{O}_{\mathcal{S}}(\mathcal{Y}_{[a^p,b^p]}), \mathcal{O}_{\mathcal{S}}^+(\mathcal{Y}_{[a^p,b^p]}))$$

and thus gives rise to an identification $\mathcal{O}_{\mathcal{Y}_{[a^p,b^p]}} \cong \mathcal{O}_{\mathcal{Y}_{[a,b]}}$. □

Remark. Moreover, we can use Proposition 3.2.15 to get a natural isomorphism $\mathcal{O}_{\mathcal{Y}} \cong \phi_* \mathcal{O}_{\mathcal{Y}}$ whose induced map on the global sections coincides with the Frobenius automorphism of B.

Proposition 3.2.19. The set $\mathcal{X}$ is naturally an adic space.

Proof. Choose an element $r \in |\mathfrak{m}_F|$ and a rational number q with $1/p < q < 1$. For every $n \in \mathbb{Z}$, we have $r^{p^n}, r^{qp^n} \in |\mathfrak{m}_F|$ as F is algebraically closed. Let us set

$$\mathcal{V}_n := \mathcal{Y}_{[r^{p^n}, r^{qp^n}]} \quad \text{and} \quad \mathcal{W}_n := \mathcal{Y}_{[r^{qp^n}, r^{p^{n-1}}]} \quad \text{for each } n \in \mathbb{Z}.$$

It is not difficult to see that each $x \in \mathcal{Y}$ lies in $\mathcal{V}_n$ or $\mathcal{W}_n$ for some $n \in \mathbb{Z}$. In addition, we find $\phi(\mathcal{V}_n) = \mathcal{V}_{n-1}$ and $\phi(\mathcal{W}_n) = \mathcal{W}_{n-1}$ for each $n \in \mathbb{Z}$ by Proposition 3.2.18. Since the sets $\mathcal{V}_n$ and $\mathcal{W}_n$ for every $n \in \mathbb{Z}$ are open adic spaces in $\mathcal{Y}$ by Proposition 3.2.14, we deduce from Proposition 3.2.10 that the action of ϕ on $\mathcal{Y}$ is properly discontinuous and consequently identify $\mathcal{X}$ as a quotient space of $\mathcal{Y}$ with an open cover given by the adic spaces $\mathcal{V}_0$ and $\mathcal{W}_0$. Moreover, we apply Proposition 3.2.18 to obtain a sheaf $\mathcal{O}_{\mathcal{X}}$ on $\mathcal{X}$ by gluing $\mathcal{O}_{\mathcal{V}_0}$ and $\mathcal{O}_{\mathcal{W}_0}$. Now the desired assertion is evident. □

Remark. A fundamental result of Kedlaya-Liu [**KL15**] yields a natural morphism of locally ringed spaces $\mathcal{X} \to X$ which induces an equivalence of categories

$$\{ \text{ vector bundles on } X \} \xrightarrow{\sim} \{ \text{ vector bundles on } \mathcal{X} \}.$$

This equivalence is an analogue of the celebrated GAGA theorem by Serre [**Ser56**] and provides powerful means to study vector bundles on the Fargues-Fontaine curve using the modern machinery in p-adic geometry developed by Scholze [**Sch12, Sch18**]. For example, the work of Birkbeck-Feng-Hansen-Hong-Li-Wang-Ye [**BFH⁺22**] and Hong [**Hon21, Hon23, Hon25**] employs this equivalence to classify vector bundles which arise as subsheaves, quotients, or extensions of given vector bundles on the Fargues-Fontaine curve.

Exercises

1. Given a p-adic field K with residue field k, show that the kernel of the extended logarithm map $\log : (\mathbb{C}_K^\flat)^\times \to B_{\mathrm{dR}}^+$ is isomorphic to $\overline{k}^\times$.

2. Let K be a p-adic field with residue field k.

(1) For every (φ, N)-module D over $K_0 = W(k)[1/p]$, prove that N_D is nilpotent.

Hint. Take an integer $i > 0$ with $N_D^i(D) = N_D^{i+1}(D)$ and prove that $N_D^i(D)$ is naturally a (φ, N)-submodule of D with a bijective monodromy operator.

(2) For every weakly admissible filtered (φ, N)-module D over K with a unique Hodge-Tate weight, prove that N_D vanishes.

Hint. If N_D is nonzero, show that $N_D(D)$ is naturally a filtered (φ, N)-submodule of D with $\deg(N_D(D)) < \deg(D)$ and $\deg^\bullet(N_D(D)) = \deg^\bullet(D)$.

3. Let us consider the basis vectors $e_1 := (1, 0)$ and $e_2 := (0, 1)$ of $\mathbb{Q}_p^{\oplus 2}$.

(1) For every $\lambda \in \mathbb{Z}_p$ and $c \in \mathbb{Q}_p$ with $\lambda \neq 0$, show that there exists a unique normally weighted filtered (φ, N)-module $D_{\lambda,c}^{\mathrm{mon}}$ over $\mathbb{Q}_p$ of rank 2 with

$$\varphi_{D_{\lambda,c}^{\mathrm{mon}}} = \begin{pmatrix} \lambda & 0 \\ 0 & p\lambda \end{pmatrix}, \quad N_{D_{\lambda,c}^{\mathrm{mon}}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{H}(D_{\lambda,c}^{\mathrm{mon}}) = \mathbb{Q}_p(ce_1 + e_2).$$

(2) Show that a filtered (φ, N)-module D over $\mathbb{Q}_p$ of rank 2 with $N_D \neq 0$ is weakly admissible if and only if it admits an isomorphism

$$D \simeq D_{\lambda,c}^{\mathrm{mon}} \otimes_{\mathbb{Q}_p} D_{\mathrm{st}}(\mathbb{Q}_p(n))$$

for some $\lambda \in \mathbb{Z}_p$, $c \in \mathbb{Q}_p$, $n \in \mathbb{Z}$ with $\lambda \neq 0$.

Hint. Represent N_D by a triangular matrix under some $\mathbb{Q}_p$-basis for D and apply the relation $N_D \circ \varphi_D = p\varphi_D \circ N_D$ to show that φ_D has two distinct $\mathbb{Q}_p$-eigenvalues.

Remark. We can combine the second part with results from Chapter III to obtain a complete classification for weakly admissible filtered (φ, N)-modules over $\mathbb{Q}_p$ of rank 2.

4. Let K be a finite extension of $\mathbb{Q}_p$.

(1) Show that every extension of $\mathbb{Q}_p$ by $\mathbb{Q}_p(1)$ is semistable.

Hint. Adapt Example 1.2.6 using the isomorphism $H^1(\Gamma_K, \mathbb{Q}_p(1)) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \widehat{K^\times}$ given by Kummer theory, where $\widehat{K^\times}$ denotes the p-adic completion of the abelian group $K^\times$.

(2) Show that every nonsplit extension of $\mathbb{Q}_p(1)$ by $\mathbb{Q}_p$ is not semistable.

Remark. The second part and Theorem 1.2.25 together imply that every nonsplit extension of $\mathbb{Q}_p(1)$ by $\mathbb{Q}_p$ is not de Rham.

5. Given a p-adic field K, verify Theorem 1.2.25 for one-dimensional p-adic Γ_K-representations.

6. For a p-adic field K with residue field k, show that every semilinear Γ_k-module M over $\widehat{K^{\mathrm{un}}}$ admits a natural isomorphism

$$M \cong M^{\Gamma_k} \otimes_K \widehat{K^{\mathrm{un}}}.$$

7. Let E be a perfect field of characteristic p and denote by φ the Frobenius automorphism on $\mathscr{E} := W(E)[1/p]$.

(1) Given integrally p-adic Γ_E-representations M and M', establish a canonical isomorphism of φ-modules

$$D_{\mathrm{int}}(M \otimes_{\mathbb{Z}_p} M') \cong D_{\mathrm{int}}(M) \otimes_{\mathcal{O}_{\mathscr{E}}} D_{\mathrm{int}}(M').$$

(2) Given a free integrally p-adic Γ_E-representation M with dual $M^\vee$, establish a canonical perfect pairing of φ-modules

$$D_{\mathrm{int}}(M) \otimes_{\mathcal{O}_{\mathscr{E}}} D_{\mathrm{int}}(M^\vee) \longrightarrow \mathcal{O}_{\mathscr{E}}.$$

(3) Given a torsion integrally p-adic Γ_E-representation M with Pontryagin dual $M^\wedge$, establish a canonical perfect pairing

$$D_{\mathrm{int}}(M) \otimes_{\mathcal{O}_{\mathscr{E}}} D_{\mathrm{int}}(M^\wedge) \longrightarrow \mathscr{E}/\mathcal{O}_{\mathscr{E}}$$

which is compatible with the φ-endomorphisms.

8. Let E be a perfect field of characteristic p and denote by φ the Frobenius automorphism on $\mathscr{E} := W(E)[1/p]$.

(1) For every étale φ-module D over $\mathscr{E}$, show that φ_D^{lin} is an isomorphism.

(2) Find a φ-module D over $\mathscr{E}$ which is not étale with φ_D^{lin} being an isomorphism.

9. Let φ denote the Frobenius automorphism on $E_{\mathbb{Q}_p}$.

(1) For every $c \in \mathbb{F}_p^\times$, prove that there exists a unique (φ, Γ_∞)-module D_c^{un} over $E_{\mathbb{Q}_p}$ of rank 1 with $\varphi_{D_c^{\mathrm{un}}} = c\varphi$ and the trivial Γ_∞-action.

(2) Prove that every (φ, Γ_∞)-module D over $E_{\mathbb{Q}_p}$ of rank 1 admits an isomorphism

$$D \simeq D_c^{\mathrm{un}} \otimes_{\mathbb{F}_p} E_{\mathbb{Q}_p}(n)$$

for some unique $c \in \mathbb{F}_p^\times$ and $n \in \mathbb{Z}$ with $0 \leq n \leq p-2$.

Hint. The field $E_{\mathbb{Q}_p}$ is isomorphic to the t-adic completion $\mathbb{F}_p((t^{1/p^\infty}))$ of $\mathbb{F}_p(t^{1/p^\infty})$, where t^{1/p^∞} denotes the set of p-power roots of the variable t, with the Γ_∞-action given by the relation $\gamma(t) = (1+t)^{\chi_{\mathbb{Q}_p}(\gamma)}$ for every $\gamma \in \Gamma_\infty$.

Remark. The second part and Theorem 2.2.14 together imply that there exist precisely $(p-1)^2$ isomorphism classes of 1-dimensional mod-p $\Gamma_{\mathbb{Q}_p}$-representations, which we can also deduce from the class field theory.

10. Let E denote the field $k((t))$ for a perfect field k of characteristic p.

(1) Show that the p-adic completion $C(E)$ of $W(k)((t))$ is a complete discrete valuation ring with residue field E and uniformizer p.

(2) Show that the Frobenius endomorphism on E lifts to an endomorphism on $C(E)$.

11. In this exercise, we study the topological space $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$ with the discrete topology on the ring $\mathbb{Z}$.

(1) Find a representative for each point on $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$.

(2) Find the closure of each point on $\mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$.

(3) Show that the image of the natural map $\mathrm{Spa}(\mathbb{Q}, \mathbb{Q}) \to \mathrm{Spa}(\mathbb{Z}, \mathbb{Z})$ induced by the embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is homeomorphic to $\mathrm{Spec}\,(\mathbb{Z})$.

12. Let (R, R^+) be a Huber pair.

(1) Prove that there exists a natural continuous map $\mathrm{Spa}(R, R^+) \to \mathrm{Spec}\,(R)$ which sends each $x \in \mathrm{Spa}(R, R^+)$ with a representative v to $v^{-1}(0)$.

(2) If R is discrete, prove that (R, R^+) is sheafy.

Hint. Prove that the structure presheaf on $\mathrm{Spa}(R, R^+)$ agrees with the pullback of the structure sheaf on $\mathrm{Spec}\,(R)$ along the map $\mathrm{Spa}(R, R^+) \to \mathrm{Spec}\,(R)$.

Remark. If R is discrete with $R = R^+$, it is not hard to see that the fibers of the natural map $\mathrm{Spa}(R, R) \to \mathrm{Spec}\,(R)$ are homeomorphic to Riemann-Zariski spaces.

13. Let F be an algebraically closed perfectoid field of characteristic p.

(1) Given an untilt C of F in characteristic 0, show that the Fontaine map θ_C induces a natural continuous map $\mathrm{Spa}(\mathcal{O}_C, \mathcal{O}_C) \to \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ with closed image.

Hint. Show that the image of the map contains an element $x \in \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ if and only if x vanishes on $\ker(\theta_C)$.

(2) Show that every classical point on $\mathcal{Y} = \mathcal{Y}_F$ is closed.

Hint. Identify every classical point on $\mathcal{Y}$ as the preimage of $\mathcal{Y}$ under the continuous map $\mathrm{Spa}(\mathcal{O}_C, \mathcal{O}_C) \to \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}})$ for some untilt C of F in characteristic 0.

14. Let F be an algebraically closed perfectoid field of characteristic p.

(1) Show that there exists a canonical embedding $(0, 1) \hookrightarrow \mathcal{Y} = \mathcal{Y}_F$ which sends each $\rho \in (0, 1)$ to the equivalence class of the Gauss ρ-norm on $A_{\mathrm{inf}} = A_{\mathrm{inf}}(F)$.

(2) Show that the image of the embedding $(0, 1) \hookrightarrow \mathcal{Y}$ is disjoint from the set of classical points on $\mathcal{Y}$.

(3) Given two elements a, b in the value group of F with $a, b < 1$, show that the embedding $(0, 1) \hookrightarrow \mathcal{Y}$ restricts to an embedding $[a, b] \hookrightarrow \mathcal{Y}_{[a,b]}$.

15. Let F be an algebraically closed perfectoid field of characteristic p.

(1) Prove that the Frobenius action ϕ on $\mathcal{Y} = \mathcal{Y}_F$ restricts to the Frobenius action on the set $Y = Y_F$ of equivalence classes of untilts of F in characteristic 0.

(2) Prove that $\mathcal{Y}$ admits a natural isomorphism $\mathcal{O}_{\mathcal{Y}} \cong \phi_* \mathcal{O}_{\mathcal{Y}}$ whose induced map on the global sections coincides with the Frobenius automorphism on $B = B_F$.

(3) Prove that $\mathcal{X} = \mathcal{X}_F$ admits a canonical isomorphism $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) \cong \mathbb{Q}_p$.

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